

Square matrices with non-negative entries whose columns sum to one are Markov.

1. Over a period of time corresponding to a generation, and within a particular geographical region, observations are made of the population movements among people living in rural areas, towns, and cities:
 - (a) 60% of rural areas, 20% of towns, 20% of cities, move to the rurals
 - (b) 20% of rural areas, 40% of towns, 20% of cities, move to the towns
 - (c) 20% of rural areas, 40% of towns, 60% of cities, move to the cities.

If at a particular time the number of people in rural areas is 44,000 and the population level then in towns is 24,000 while the numbers living in cities is 32,000 at that time, calculate the population levels to be expected in *rural areas*, *towns*, and *cities* after a time period of a generation has passed.

2. The following matrix has elements formed from the percentages given above. Its rows and columns are labelled in the order *rural areas*, *towns*, and *cities*. Show that the product of the matrix with the column vector of population values, in units of a thousand for convenience, reproduces in the correct order the population levels you have just evaluated.

$$M \begin{bmatrix} 44 \\ 24 \\ 32 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 6 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} 44 \\ 24 \\ 32 \end{bmatrix}.$$

3. Verify by computing $M M^{-1}$ that the inverse of the Markov matrix M is as given just on the right of the equals sign in the following equation.

$$M^{-1} \begin{bmatrix} 44 \\ 24 \\ 32 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 4 & -1 & -1 \\ -2 & 8 & -2 \\ 0 & -5 & 5 \end{bmatrix} \begin{bmatrix} 44 \\ 24 \\ 32 \end{bmatrix}$$

4. Calculate the product shown to obtain the set of population levels in the *rural areas*, *towns*, and *cities* of the previous generation.
5. Show that multiplication of the column vector for the previous generation by the Markov matrix M provides the set of population levels in the the original generation, namely 44, 24, 32, thousand in the rural areas, towns, and cities, respectively.
6. Briefly describe the way in which you would evaluate the three population levels two generations onwards, three generations onwards, and so on.
7. Indicate how you would determine the limiting populations, if they exist, after many generations have passed.

For a discussion of Markov Chains see Chapter 5 of *Contemporary Linear Algebra* by Howard Anton and R. C. Busby, John Wiley, (www.wiley.com/college/anton).

Observe that the Markov matrix introduced above may be written in various forms

$$M = \frac{1}{5} \begin{bmatrix} 3 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.2 & 0.4 & 0.2 \\ 0.2 & 0.4 & 0.6 \end{bmatrix}.$$

Note that the matrix entries are non-negative and that the columns sum to unity in this square probability matrix. It may be shown that the inverse of the Markov matrix M is

$$M^{-1} = \frac{1}{2} \begin{bmatrix} 4 & -1 & -1 \\ -2 & 8 & -2 \\ 0 & -5 & 5 \end{bmatrix}$$

by verifying that the following is true for the probability matrix M and its inverse M^{-1}

$$M M^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and that the following is correct indicating that matrix M commutes with its inverse

$$M^{-1} M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where the unit three-by-three matrix is displayed.

In the above exercise, suppose that after many generations the number of thousands of people in *rural areas*, *towns*, and *cities*, say x , y and z respectively, does not change. Would the following equation be valid?

$$M \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

And would it be true that

$$x + y + z = 44 + 24 + 32?$$

Recall that the sum of the three population levels in each succeeding generation is constant in time, a property of the total population that is induced by the probability nature of the matrix M where the columns of non-negative matrix entries sum to one.