Tutorial on Eigenvalues 1S2 JF Natural Sciences

Consider the three matrices each with two rows and two columns

$$A = \begin{bmatrix} 5 & 1 \\ 2 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} 6 & 4 \\ -1 & 2 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & -2 \\ 5 & 3 \end{bmatrix}.$$

1. Determine the characteristic equation of the matrix A, i.e., the polynomial equation for its eigenvalues.
2. Find the eigenvalues of the matrix A and indicate if they are real or imaginary and also if they are equal or unequal.
3. Find the eigenvalues of the matrix B and indicate if they are real or imaginary and also if they are equal or distinct.
4. Find the eigenvalues of the matrix C and indicate if they are real or imaginary and also if they are equal or different.
5. For each eigenvalue of the matrix A obtain the corresponding eigenvector of the matrix A.
6. Verify that each eigenvector V of A, and its corresponding eigenvalue λ, obeys the matrix equation
$$A\,V = \lambda\,V$$
indicating that the vector $A\,V$ is parallel to the vector V, viewing the result geometrically.
7. By contrast, show that column vector $A\,U$, where U is the transpose of the row vector $(\,0 \quad 1\,)$, is not a multiple of the vector U and does not have the same direction in a geometric interpretation.
8. Evaluate A^2, the square of the matrix A.
9. Show that the matrix A obeys its characteristic equation when λ is replaced by A. (The Cayley-Hamilton theorem).

See Section 4.4 of *Contemporary Linear Algebra* by Howard Anton and Robert C. Busby (Wiley), or the end of Section 2.3 of *Elementary Linear Algebra & Applications* by Howard Anton and Chris Rorres (Wiley).

BACKGROUND MATERIAL FOR EIGENVALUES OF A MATRIX

The eigenvalues and corresponding eigenvectors of the following square matrix

$$A = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$$

are obtained from the condition that the defining equation for an eigenvalue λ

$$A\,V \,=\, \lambda\,V \qquad \text{or} \qquad (A \,-\, \lambda\,I)\,V \,=\, 0$$

has non trivial solutions for the vector V, namely the determinantal condition

$$\det(A \,-\, \lambda\,I) \,=\, 0$$

or

$$\begin{vmatrix} 4-\lambda & 3 \\ 2 & 5-\lambda \end{vmatrix} = 0 \qquad \text{or} \qquad (4-\lambda)\,(5-\lambda) \,-\, 6 \,=\, 0$$

and therefore the characteristic equation, polynomial in λ, of the matrix A reads

$$\lambda^2 \,-9\,\lambda \,+\, 14 \,=\, 0$$

leading to the eigenvalues of the matrix A, $\lambda = 2$ and $\lambda' = 7$. The characteristic polynomial for the matrix A is known to be zero by the Cayley-Hamilton theorem

$$A^2 \,-9\,A \,+\, 14 \,=\, 0\,.$$

The eigenvectors of A are obtained from the above equation $(A \,-\, \lambda\,I)\,V \,=\, 0$ for each of the eigenvalues, that is, from the equation

$$\begin{bmatrix} 4-\lambda & 3 \\ 2 & 5-\lambda \end{bmatrix} \begin{bmatrix} u \\ w \end{bmatrix} = 0$$

and writing out this matrix equation in component form provides the eigenvector V of A corresponding to the eigenvalue $\lambda = 2$ as follows

$$V \,=\, \begin{bmatrix} u \\ w \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

when we select $u = 3$ and $w = -2$ to obey $(A \,-\, \lambda\,I)\,V \,=\, 0$ or $A\,V \,=\, \lambda\,V$. The second eigenvector V' corresponding to the eigenvalue $\lambda' = 7$ is the vector

$$V' \,=\, \begin{bmatrix} u' \\ w' \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

when we select $u' = 1$ and $w' = 1$ to obey $(A \,-\, \lambda'\,I)\,V' \,=\, 0$ or $A\,V' \,=\, \lambda'\,V'$ again written out in component form.