The following matrix with rows of elements summing to unity is Markov for $n > 0$

$$M \quad = \quad \frac{1}{n} \begin{bmatrix} n-a-b & a & b \\ a & n-2a-c & a+c \\ c & a & n-a-c \end{bmatrix} .$$

Values of a, b, c, are such that, $0 \le a$, $0 \le b$, $0 \le c$, $a+b \le n$, $2a+c \le n$, and are often assigned to integers in examples. The case $n = 10$ provides convenient decimal fractions. The Perron Frobenius theorem indicates that there is a unit eigenvalue of a Markov matrix and so the eigenvalues of M are

$$1\,, \qquad \lambda = 1-(a+b+c)/n\,, \qquad \lambda' = 1-(3a+c)/n\,.$$

In the case of unequal eigenvalues, $b-2a \neq 0$, the corresponding right eigenvectors of the Markov matrix M, appropriately normalised, form the columns of the matrix

$$P \;=\; \frac{(b-2a)^{-1}}{a+b+c} \begin{bmatrix} b-2a & (b-2a)(a+b) \;\;+\;\; a(a-c) & -a(a+b+c) \\ b-2a & (b-2a)(-a)-(2a+c)(a-c) & (2a+c)(a+b+c) \\ b-2a & (b-2a)(-c) \;\;+\;\; a(a-c) & -a(a+b+c) \end{bmatrix}$$

and the corresponding left eigenvectors of M form the rows of the inverse matrix

$$P^{-1} \;=\; \frac{1}{3a+c} \begin{bmatrix} (a+c)^2 & a(b-2a)+a(3a+c) & (b-2a)(2a+c)+a(5a+3c) \\ 3a+c & 0 & -(3a+c) \\ a-c & b-2a & -(b-2a)-(a-c) \end{bmatrix}$$

so that the matrix M may be diagonalised with eigenvalues being on the diagonal

$$P^{-1}\,M\,P \quad = \quad \frac{1}{n} \begin{bmatrix} n & 0 & 0 \\ 0 & n-a-b-c & 0 \\ 0 & 0 & n-3a-c \end{bmatrix} .$$

In the case, $b-2a = 0$, the two non-leading eigenvalues are equal, and the matrix

$$M = \frac{1}{n} \begin{bmatrix} n-3a & a & 2a \\ a & n-2a-c & a+c \\ c & a & n-a-c \end{bmatrix}$$

has only two linearly independent eigenvectors, the second and third columns of the matrix P becoming proportional, as occurs with the second and third rows of the matrix P^{-1}. The two right eigenvectors appear in the first and second columns of

$$P \quad = \quad \frac{1}{3a+c} \begin{bmatrix} a & -a & 3a-at \\ a & 2a+c & -a+(2a+c)t \\ a & -a & -c-at \end{bmatrix} .$$

Freedom in the choice of the third column is characterised by an arbitrary parameter t whose value should be the same as that appearing in the second row of the inverse matrix, in which the equilibrium left eigenvector of M and its non-leading eigenvector form the first and third rows. For convenience $d = 3a + c$ is introduced

$$P^{-1} \quad = \quad \frac{1}{d}\begin{bmatrix} a + 2c + c^2/a & d & 5a + 3c \\ a - c - dt & d & -4a + dt \\ d & 0 & -d \end{bmatrix},$$

The Markov matrix M is not diagonisable if $a \neq c$ (again in the case $b = 2a$) and may be rendered in Jordan form through the similarity transformation

$$P^{-1}\, M\, P \quad = \quad \frac{1}{n}\begin{bmatrix} n & 0 & 0 \\ 0 & n-d & (a-c)(1+at/d) \\ 0 & 0 & n-d \end{bmatrix}.$$

In the case where $t = 0$, an integer power k of the matrix M may be evaluated from

$$\frac{M^k}{d^{-2}} = \begin{bmatrix} a & -a & 3a \\ a & 2a+c & -a \\ a & -a & -c \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda^k & k(a-c)\lambda^{k-1}/n \\ 0 & 0 & \lambda^k \end{bmatrix}\begin{bmatrix} (a+c)^2/a & d & 5a+3c \\ a-c & d & -4a \\ d & 0 & -d \end{bmatrix}$$

where the eigenvalues are 1 and $\lambda = 1 - d/n$ (twice).

In the case where $b - 2a = 0$ and $a = c$, the Markov matrix M takes the form

$$M \quad = \quad \frac{1}{n}\begin{bmatrix} n-3a & a & 2a \\ a & n-3a & 2a \\ a & a & n-2a \end{bmatrix}$$

which may be diagonalised by the following simpler matrix P and its inverse P^{-1}

$$P = \frac{1}{2}\begin{bmatrix} 1 & -1 & 3 \\ 1 & 3 & -1 \\ 1 & -1 & -1 \end{bmatrix}, \qquad P^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix}$$

the three eigenvalues appearing on the diagonal being 1, $1 - 4a/n$, and $1 - 4a/n$.

A connected Markov chain is reversible if its equilibrium eigenvector v_i satisfies

$$v_i\, M_{i\,j} \;=\; v_j\, M_{j\,i} \quad \text{for each } i,\ j = 1,\ 2,\ 3\,,$$

and reversibility in the example above is ensured by the condition, $ab = c(a+c)$. A reversible Markov matrix can be symmetrised by a similarity transformation that uses a diagonal matrix formed from the square roots of elements of its equilibrium eigenvector, together with the inverse diagonal matrix. Values $a = 1$, $b = 12$, $c = 3$, and $n = 20$, provide an example of the symmetrisation of $20\,M$ whose equilibrium left eigenvector, according to the first row of the matrix P^{-1} above, is (16, 16, 64)

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 8 \end{bmatrix}\begin{bmatrix} 7 & 1 & 12 \\ 1 & 15 & 4 \\ 3 & 1 & 16 \end{bmatrix}\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 8 \end{bmatrix}^{-1} = \begin{bmatrix} 7 & 1 & 6 \\ 1 & 15 & 2 \\ 6 & 2 & 16 \end{bmatrix}.$$

The resulting matrix may, of course, be diagonalised in the usual way.

Sample Markov matrices are exhibited, but for a factor of $n = 10$, together with the matrices that diagonalise them, or bring them to Jordan form. In the first matrix, the first row is the equilibrium eigenvector of M while the following rows provide other left eigenvectors. The matrix in the middle on the left is $10\,M$. The columns of the third matrix include the right eigenvectors of M. The diagonal matrix on the right hand side includes the eigenvalues of M scaled up by a factor of $n = 10$.

The selection $a = 1$, $b = 4$ and $c = 2$, provides a matrix with unequal eigenvalues

$$\frac{1}{5}\begin{bmatrix} 9 & 7 & 19 \\ 5 & 0 & -5 \\ -1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 5 & 1 & 4 \\ 1 & 6 & 3 \\ 2 & 1 & 7 \end{bmatrix}\begin{bmatrix} 2 & 9 & -7 \\ 2 & 2 & 28 \\ 2 & -5 & -7 \end{bmatrix}\frac{1}{14} = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

The choice $a = 1$, $b = 2$, $c = 1$, gives a Markov matrix with equal sub-leading eigenvalues resulting from $b = 2a$ that is furthermore diagonalisable since $a = c$,

$$\frac{1}{2}\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 7 & 1 & 2 \\ 1 & 7 & 2 \\ 1 & 1 & 8 \end{bmatrix}\begin{bmatrix} 1 & -1 & 3 \\ 1 & 3 & -1 \\ 1 & -1 & -1 \end{bmatrix}\frac{1}{2} = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}.$$

The selection $a = 1$, $b = 2$, $c = 2$, gives a Markov matrix with equal sub-leading eigenvalues induced by $b = 2a$, and is not diagonalisable since $a \neq c$,

$$\frac{1}{5}\begin{bmatrix} 9 & 5 & 11 \\ -1 & 5 & -4 \\ 5 & 0 & -5 \end{bmatrix}\begin{bmatrix} 7 & 1 & 2 \\ 1 & 6 & 3 \\ 2 & 1 & 7 \end{bmatrix}\begin{bmatrix} 1 & -1 & 3 \\ 1 & 3 & -1 \\ 1 & -1 & -2 \end{bmatrix}\frac{1}{5} = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 5 & -1 \\ 0 & 0 & 5 \end{bmatrix}.$$

The selection $a = 2$, $b = 4$, $c = 1$, gives a Markov matrix with equal sub-leading eigenvalues caused by $b = 2a$, that again is not diagonalisable since $a \neq c$,

$$\frac{1}{7}\begin{bmatrix} 9 & 14 & 26 \\ 1 & 7 & -8 \\ 7 & 0 & -7 \end{bmatrix}\begin{bmatrix} 4 & 2 & 2 \\ 2 & 5 & 3 \\ 1 & 2 & 7 \end{bmatrix}\begin{bmatrix} 1 & -2 & 6 \\ 1 & 5 & -2 \\ 1 & -2 & -1 \end{bmatrix}\frac{1}{7} = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}.$$

Values $a = 1$, $b = 2$, $c = 1$, $n = 10$, indicate an example of such a matrix with equal eigenvalues where reversibility ensures that it is diagonalisable, that is, $a = c$.

Values $a = 1$, $b = 6$, $c = 2$, $n = 10$, give another example of a reversible Markov matrix whose transpose and leading eigenvector is shown. A re-arranged form is in the second example. Values $a = 1$, $b = 2$, $c = 1$, $n = 10$, indicate an example of such a matrix with equal eigenvalues where reversibility ensures that it is diagonalisable, that is, $a = c$. Again, the transpose and leading eigenvector is exhibited.

$$\frac{1}{10}\begin{bmatrix} 3 & 1 & 2 \\ 1 & 6 & 1 \\ 6 & 3 & 7 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \quad \frac{1}{10}\begin{bmatrix} 6 & 1 & 1 \\ 3 & 7 & 6 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \quad \frac{1}{10}\begin{bmatrix} 7 & 1 & 1 \\ 1 & 6 & 1 \\ 2 & 3 & 8 \end{bmatrix}\begin{bmatrix} 9 \\ 4 \\ 8 \end{bmatrix}.$$