Inverse Matrices — JF Life and Earth Sciences

Square matrices with non-negative entries whose columns sum to one are Markov.

1. Over a period of time corresponding to a generation, and within a particular geographical region, observations are made of the population movements among people living in rural areas, towns, and cities:

   (a) 60% of rural areas, 20% of towns, 20% of cities, move to the rurals

   (b) 20% of rural areas, 40% of towns, 20% of cities, move to the towns

   (c) 20% of rural areas, 40% of towns, 60% of cities, move to the cities.

   If rows and columns label *rural areas, towns, and cities* in that order, explain why the square matrix M describing the changes in population over one generation is the Markov matrix

$$M = \frac{1}{10}\begin{bmatrix} 6 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 4 & 6 \end{bmatrix}.$$

2. Rural, town and city numbers may be calculated in successive generations through repeated multiplication by the Markov matrix M. Obtain the population in the rural areas, towns, and cities, r_3, t_3, c_3, in generation $n = 3$ by calculating the product of the matrix M with the column vector of population values in units of a thousand, $r_2 = 44$, $t_2 = 24$, $c_2 = 32$, corresponding to generation $n = 2$, according to

$$\begin{bmatrix} r_{n+1} \\ t_{n+1} \\ c_{n+1} \end{bmatrix} = M \begin{bmatrix} r_n \\ t_n \\ c_n \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 3 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} r_n \\ t_n \\ c_n \end{bmatrix}.$$

3. Find the inverse, M^{-1}, of the matrix M by using a row reduction method on the matrix M augmented by the unit 3×3 matrix. Calculate $M^{-1}M$.

4. Use the inverse to calculate the thousands of people in the rural areas, towns, and cities, r_1, t_1, c_1, for generation $n = 1$, given the above population values of generation $n = 2$,

$$\begin{bmatrix} r_{n-1} \\ t_{n-1} \\ c_{n-1} \end{bmatrix} = M^{-1} \begin{bmatrix} r_n \\ t_n \\ c_n \end{bmatrix}.$$

5. Obtain the initial values of the populations of the rural areas, towns and cities in generation $n = 0$. Indicate how you would determine the limiting populations after many generations have passed.

For a discussion of Markov Chains see Chapter 5 of *Contemporary Linear Algebra* by Howard Anton and R. C. Busby, John Wiley, (`www.wiley.com/college/anton`).

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