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Hyperbolic Dimension for Interval Maps

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Abstract. The hyperbolic and Hausdorff dimensions are shown to coincide for C^2 maps without recurrent critical points. The maps may have parabolic periodic points. The Julia set for certain such maps may have hyperbolic dimension equal to 1 but Lebesgue measure equal to 0.

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1. Introduction

The coincidence of different notions of fractal dimensions has been a topic of interest for some time. The hyperbolic dimension is defined as the supremum of Hausdorff dimensions of hyperbolic subsets. In general the dynamics of hyperbolic sets is well understood. If Hausdorff and hyperbolic dimensions coincide then one may say that the hyperbolic subsets approximate, in some sense, the original set. We study this problem in the cadre of C^2 maps of an interval I.

Definition 1.1 *An interval map $f : I \to I$ is* of class D *(we also say f is a* D-map*) if*

- *f is of class C^2;*
- *the critical set $Crit = \{c \in I : Df(c) = 0\}$ is of finite cardinality;*
- *there are at most a finite number of parabolic periodic points of any one period;*
- *all critical points are C^2* non-flat*, i.e. for each $c \in Crit$, there is a C^2 diffeomorphism ψ and an $l > 1$ such that locally $f(x) = \pm|\psi(x-c)|^l + f(c)$;*
- *$f : \partial I \to \partial I$.*

One can define the Fatou set of an interval map as the set of points for which there is a neighbourhood on which iterates of f form an equicontinuous family. For C^2 maps this coincides with the interior of the basins of attraction of periodic attractors. The *Julia set* $\mathcal{J}$ is the complement of the Fatou set. See [1] for details.

Given a continuous interval map $f : I \to I$ and subset $K \subset I$, let $\mathcal{W}$ be the collection of connected components of $I \setminus K$. Following [2] we define the collection of sets $\tau(K)$ as follows:

$$\tau(K) = \{W \in \mathcal{W} : f(W) \cap K \neq \emptyset\}.$$

Definition 1.2 *Given a C^1 interval map $f : I \to I$ we say K is a τ-set for f if*

- *K is a forward-invariant (i.e. $f(K) \subset K$) compact subset of the Julia set of f;*
- *the cardinality of $\tau(K)$ is finite;*
- *for all $W \in \tau(K)$, ∂W is pre-periodic.*

The *omega-limit set* $\omega(x)$ of a point x is defined by

$$\omega(x) = \bigcap_{N>0} \overline{\{f^n(x) : n \geq N\}}$$

and that of a set S by

$$\omega(S) = \bigcup_{x \in S} \omega(x).$$

A point x for D-map f is *non-recurrent* if $x \notin \omega(x)$.

Theorem 1 *Let K be a τ-set for a D-map f such that each critical point of f contained in K is non-recurrent. Then the Hausdorff and hyperbolic dimensions of K coincide.*

Corollary 1.3 *Suppose $f : I \to I$ is a D-map such that each critical point is non-recurrent. Then the Hausdorff and hyperbolic dimensions of the Julia set $\mathcal{J}$ of f coincide, i.e.*

$$HD(\mathcal{J}) = HypD(\mathcal{J}).$$

Proof: The Julia set is completely invariant so $\tau(\mathcal{J}) = \emptyset$ and $\mathcal{J}$ is a τ-set. Apply the theorem to conclude. $\square$

Definition 1.4 *A parabolic periodic point p of period q for interval map f shall be called* flat *if one can write*

$$f^{2q}(x) = x + \psi(x),$$

where $\lim_{x \to p} \psi(x)(x-p)^{-n} = 0$ *for all $n \geq 0$.*

For C^∞ maps this condition on ψ is equivalent to $D^n\psi(p) = 0$ for all $n \geq 0$.

Theorem 2 *Suppose for a D-map f there exists a closed interval U with the following properties:*

- *there exists a flat parabolic point of period q such that $p \in \partial U$;*
- *U contains no critical points of f^{2q};*
- *$\frac{f^{2q}(x)-p}{x-p} > 1$ for all $x \in U$;*
- *there exists $V = [y, f^{2q}(y)]$ contained in U and an integer n such that $f^n(V) \supset U$.*

Then the hyperbolic dimension of the Julia set of f is equal to 1.

Corollary 1.5 *There exists D-maps such that HypD($\mathcal{J}$) = 1 but $\lambda(\mathcal{J}) = 0$, where λ denotes Lebesgue measure.*

Of course, the dynamics away from a forward-invariant compact set does not affect the dynamics on the set itself. This motivates the following definition.

Definition 1.6 *Let K be a compact set of an interval I. Two maps f and g are K-*equivalent *if there is a neighbourhood V of K such that $f_{|V} = g_{|V}$.*

Lemma 1.7 *Let f and g be K-equivalent maps of class C^1. Then the hyperbolic dimension of K with respect to f and the hyperbolic dimension of K with respect to g coincide.*

Proof: A set $K' \subset K$ is hyperbolic for f if and only if K' is also hyperbolic for g. □

One can thus look for equivalent maps which are the best-behaved away from the set of interest K.

Definition 1.8 *Let K be a compact subset contained in the interior of an interval I. For integers $n \geq 0$ we say f is* of class $M_n(K)$ *if*

- *f is a D-map of I;*
- *K is a τ-set for f;*
- *there are exactly n critical points in K;*
- *each critical point in K is non-recurrent;*
- *all critical $c \in (I \setminus K)$ satisfy $f(c) \in \partial I$;*
- *both $x \in \partial I$ are hyperbolic, attracting, fixed points.*

The following result is slightly more general than Theorem 1 which one recovers by applying Lemma 2.1.

Theorem 3 *Given a compact subset K of an interval, let f be K-equivalent to a map of class $M_n(K)$ for some $n \geq 0$. Then the Hausdorff and hyperbolic dimensions of K coincide.*

We denote by HypD(K) the hyperbolic dimension of K. The Hausdorff dimension [7] of a set K, denoted here as HD(K), is defined as follows:

$$\mathrm{HD}(K) = \inf\{t \geq 0 : \lim_{\varepsilon \searrow 0}(\inf_{\mathcal{U}} \sum_{i=1}^{\infty}(\mathrm{diam} U_i)^t) < \infty\},$$

where $\mathcal{U} = \{U_i\}_{i=1}^{\infty}$ is a countable cover of K by sets U_i of diameter less than ε.

We say a set K is *hyperbolic* if it is forward-invariant and compact and there exists an iterate of f such that $|Df^n_{|K}| > 1$. A point p of period q is *parabolic* if and only if $|Df^q(p)| = 1$.

In [6] the *Mañé Hyperbolicity Theorem* is proven: for D-maps, if all periodic points of a forward-invariant compact K are hyperbolic repelling, with K not containing critical points, then K is hyperbolic.

In the real setting, Urbańksi [10] and Hofbauer [4] have studied weakly expanding systems (systems f, K such that $|Df|_{|K} \geq 1$ with equality only at a finite number of fixed points) in some detail. We allow the presence of critical points. Even without critical points in K, Theorem 1 treats a wider class of dynamical systems: for example, if set K contains a parabolic fixed point p for f and some point $x \in K$ satisfies $f(x) = p$ and $|Df(x)| < 1$ then no iterate of $f_{|K}$ is weakly expanding since $|Df^n(x)| = |Df(x)| < 1$. Both [10] and [4] also have extra conditions on behaviour near parabolic points.

For a survey of results about dimensions of Julia sets of rational maps of the Riemann sphere see [11]. Theorem 4.5 of [11] states that for rational maps with parabolic points and without recurrent critical points the Hausdorff and hyperbolic dimensions of the Julia set coincide. The bulk of the work was carried out in [9]. For the same class of rational maps, it is proven in [8] that either the Julia set is the whole Riemann sphere or its upper Box dimension is < 2. We show that the analagous result is not true for real maps: there exist Julia sets of Lebesgue measure zero but of hyperbolic dimension equal to 1, see Corollary 1.5.

If f is a D-map with two fixed points p and q without a critical point inbetween, then all points in $[p, q] \cap \mathcal{J}$ are fixed and repelling on at least one side. By definition of D-maps, the number of parabolic fixed points is at most countable. The number of repelling fixed points is at most countable since f is C^2. Thus $\mathrm{HD}([p, q] \cap \mathcal{J}) = 0$.

On the other hand, allowing an uncountable number of parabolic points can lead to strange Julia sets. Given any Cantor set in an interval, one can define a monotone increasing C^2 function g such that $\frac{1}{2} < Dg < 2$, g leaves the Cantor set fixed and each point of the Cantor set is parabolic with respect to g and attracting on one side and repelling on the other. If there are no other fixed points in the interior of the interval then the Cantor set coincides with the Julia set of g, yet there are no hyperbolic subsets.

The condition $f : \partial I \to \partial I$ is standard and ensures that branches of our induced maps are full. Merely taking ∂I pre-periodic is sufficient: one can extend f to a larger interval and add an element to $\tau(K)$.

The Julia set $\mathcal{J}$ of a function is completely invariant and so $\tau(\mathcal{J})$ is empty. If the Julia set contains recurrent critical points, upon removing all inverse images of suitable small neighbourhoods of the recurrent critical points, a τ-set disjoint from the recurrent critical points remains. Similarly, if one subsequently removes all inverse images of an appropriate neighbourhood of parabolic points and any remaining critical points, then a hyperbolic τ-set remains. Any hyperbolic subset of $\mathcal{J}$ is contained in such a set.

The next section contains a miscellany of results and definitions used in the rest of the paper. Section 3 uses conformal measures to give a dimension estimate for expanding induced maps. Then come sections giving dimension estimates for parabolic and critical dynamics. Section 6 ties these results together to give a short proof of Theorem 3 which combined with Lemma 2.1 proves Theorem 1. Finally in Section 7 we prove Theorem 2 and show how to construct a map which satisfies the claim of Corollary 1.5.

2. Preliminaries

Lemma 2.1 *Suppose K is a τ-set for D-map f, and that all critical points in K are non-recurrent. Let $n \geq 0$ be the number of critical points in K. Then there exists a map g of class $M_n(K)$ such that g is K-equivalent to f.*

Proof: Let V be finite open cover of K such that $\overline{V} \cap (\mathrm{Crit}(f) \setminus K) = \emptyset$. Working on a larger interval if necessary, by modifying f outside of V one can obtain a D-map g which is K-equivalent to f and which sends all critical points not in K to the hyperbolically attracting boundary.

It remains to show that K is a τ-set for g, which reduces to showing that for all $W \in \tau_g(K)$, ∂W is preperiodic (here $\tau_g(K)$ is $\tau(K)$ for the map g). Suppose there exists a positive n such that $f^n(W) \cap K \neq \emptyset$. Then $f^{n-1}(W) \in \tau_f(K)$. Also $f^{n-1} : \partial W \to \partial f^{n-1} W$ by forward invariance of K, so ∂W is preperiodic. Otherwise, for all n, $f^n(W) \cap K = \emptyset$. The absence of wandering intervals and forward invariance of K then imply that the size of iterates $|f^n(W)|$ is bounded away from 0 so ∂W must be preperiodic. □

The following is proved in [3]:

Fact 2.2 *Köbe principle: Let I be a compact interval and $f : I \to I$ be a C^2 map with all critical points C^2 non-flat. Then there exists a continuous increasing function σ, $\sigma(0) = 0$, with the following property. If $J \subset T$ are open intervals and $n \in N$ is such that f^n is a diffeomorphism on T then, for every $x, y \in J$, we have*

$$\frac{Df^n(x)}{Df^n(y)} \geq \frac{e^{-\sigma(\max_{i=0}^{n-1} |f^i(T)|) \cdot \sum_{i=0}^{n-1} |f^i(J)|}}{(1 + \nu(f^n(J), f^n(T)))^2}, \tag{1}$$

where for open intervals $\overline{A} \subset B$, $\nu(A, B) = \frac{|A|}{dist(A, \partial B)}$.

The bounding quantity (or its inverse) in inequality (1) is called a distortion bound. Note that if the $f^i(J)$ are disjoint then the sum of the lengths of these intervals is bounded by $|I|$. They will be disjoint for first entry and first return maps:

Definition 2.3 *An interval U is* regularly returning *if and only if $f^n(\partial U) \cap U = \emptyset$ for every $n > 0$.*

Its nice property is if A, B are intervals, $f^n(A) = f^m(B) = U, n < m$ and $A \cap B \neq \emptyset$, then $B \subset A$. The *first entry time* to U, $e(x)$, for a point x is defined as

$$e(x) = \inf\{k \geq 0 : f^k(x) \in U\}.$$

The *first return time* to U, $r(x)$, for a point $x \in U$ is defined as

$$r(x) = \inf\{k \geq 1 : f^k(x) \in U\}.$$

The first entry map ψ_U for U is defined by $\psi_U(x) = f^{e(x)}(x)$ where $e(x)$ is defined. Similarly the first return map ϕ_U for U is defined as follows: $\phi_U(x) = f^{r(x)}(x)$. This is only defined where $r(x)$ is defined.

If $J \ni x$ is a connected component of the domain of ϕ_U for regularly returning interval U, then the intervals $J, f(J), \ldots, f^{r(x)-1}(J)$ are pairwise disjoint, and $r(x)$ is constant on J. This disjointedness property makes first return (and similarly first entry) maps easy to work with.

When applying the Köbe principle, to bound $\max |f^i(T)|$ one makes use of the Contraction Principle [12].

Fact 2.4 *Let f be a C^2 map with a finite number of C^2 non-flat, non-periodic critical points. Then there exists a continuous increasing function γ, $\gamma(t) \to 0$ as $t \to 0$, such that if f^n maps open interval A diffeomorphically onto B, B disjoint from immediate basins of periodic attractors, then $|A| < \gamma(|B|)$.*

The remaining $\nu(f^n(J), f^n(T))$ tends to 0 as $|f^n(J)|$ does if $f^n(J)$ stays away from $\partial f^n(T)$. In this paper $f^n(J)$ shall be a regularly returning interval compactly contained in some interval disjoint from the forward orbit of the critical set.

A differentiable map ϕ is *expanding* if $|D\phi| \geq \lambda > 1$, for some constant λ, everywhere on the domain of definition of ϕ.

Lemma 2.5 *Let I be a compact interval and $f : I \to I$ be a C^2 map with all critical points C^2 non-flat. Suppose $U \subset I$ is a regularly returning interval such that the first return map ϕ to U is expanding, and that there exists an open interval $V \supset \overline{U}$ such that every branch of ϕ extends (as an iterate of f) to a diffeomorphism onto V.*

Then there exists a constant $C > 1$ such that

$$\frac{D\phi^n(x)}{D\phi^n(y)} \leq C$$

for all x, y in the same connected component of the domain of ϕ^n.

Proof: Let J be a connected component of the domain of ϕ and n_J the number satisfying $f^{n_J}_{|J} = \phi_{|J}$. Since U is regularly returning, $\{f^i(J)\}_{i=0}^{n_J-1}$ is a pairwise disjoint collection of intervals. Hence

$$\sum_{i=k}^{n_J-1} |f^i(J)| \leq |I|$$

for all $k = 0, 1, \ldots, n_J - 1$. Applying the Köbe principle, there exists a constant $C_0 > 1$ independent of the connected component J such that

$$\frac{Df^k(x)}{Df^k(y)} \leq C_0$$

for all x, y in $f^{n_J-k}(J)$, for all $k = 1, 2, \ldots, n_J$, for all connected components J of the domain of ϕ.

Let $g : W \to g(W)$ be a diffeomorphism of an interval W with distortion bounded by C_0, i.e. such that $\frac{Dg(x)}{Dg(y)} \leq C_0$ for all x, y in W. Let L be a subinterval of W. Writing $M = \sup_W |Dg(x)|$, it follows that $|g(L)| \geq \frac{M}{C_0}|L|$ and $|g(W)| \leq M|W|$, so

$$\frac{|g(L)|}{|g(W)|} \geq \frac{|L|}{C_0|W|}.$$

Hence, if $C_0|g(L)| \leq \alpha|g(W)|$ then $|L| \leq \alpha|W|$, for any $\alpha > 0$. Thus, if J is a connected component of the domain of ϕ and L is a subinterval of J such that $C_0|\phi(L)| \leq \alpha|U|$ then

$$|f^i(L)| \leq \alpha|f^i(J)|$$

for all $i = 0, 1, \ldots, n_J$, and so

$$\sum_{i=0}^{n_J-1} |f^i(L)| \leq \alpha \sum_{i=0}^{n_J-1} |f^i(J)|. \tag{2}$$

Let $\lambda > 1$ satisfy $|D\phi| \geq \lambda$ everwhere on its domain. There is an $N > 0$ such that $\lambda^N > C_0$. Let A be a connected component of the domain of ϕ^{N+n} for some $n > 0$. Since $|D\phi| \geq \lambda$, we have

$$|\phi^k(A)| \leq \lambda^{-N}\lambda^{k-n}|U|$$

for all $k \leq N + n$. Let $J(k)$ denote the connected component of the domain of ϕ containing $\phi^{k-1}(A)$. One has $C_0|\phi(\phi^{k-1}(A))| \leq C_0\lambda^{-N}\lambda^{k-n}|U| \leq \lambda^{k-n}|U|$, so by (2), we deduce, for $k = 1, \ldots, N + n$, that

$$\sum_{i=0}^{n_{J(k)}-1} |f^i(\phi^{k-1}(A))| \leq \lambda^{k-n} \sum_{i=0}^{n_{J(k)}-1} |f^i(J(k))| \leq \lambda^{k-n}|I|.$$

Let m be such that $f^m_{|A} = \phi^{N+n}_{|A}$. It follows from the previous line that

$$\begin{aligned}\sum_{i=0}^{m-1} |f^i(A)| &\leq \sum_{k=1}^{N+n} \lambda^{k-n}|I| \\ &= \lambda^{-n}\frac{\lambda^{N+n+1} - \lambda}{\lambda - 1}|I| \\ &\leq \frac{\lambda^{N+1}}{\lambda - 1}|I| =: C_1\end{aligned}$$

say. Apply the Köbe principle one last time to get

$$\frac{Df^m(x)}{Df^m(y)} \geq \frac{e^{\sigma(|U|)C_1}}{(1 + \nu(U, V))^2} =: C^{-1}$$

say, for all x, y in A. The constant C not depending on A or on n, we conclude that for all $n > 0$,

$$\frac{D\phi^n(x)}{D\phi^n(y)} \leq C$$

for all x, y in the same connected component of the domain of ϕ^n. □

Lemma 2.6 *Let f be a C^1 map of the interval I with a finite number of critical points. For any set $J \subset I$ one has HD$(f(J))$ = HD(J).*

Proof: Let B_δ be the set of all points of I less than some $\delta > 0$ away from Crit. On B_δ the derivative of f is bounded away from 0. Any cover of $J \setminus B_\delta$ disjoint from B_δ is mapped by f to a cover of $f(J \setminus B_\delta)$ with bounded distortion. It follows that $\mathrm{HD}(J \setminus B_\delta) = \mathrm{HD}(f(J \setminus B_\delta))$ for all $\delta > 0$. Write $J = \bigcup_{\delta>0}(J \cap B_\delta) \cup \mathrm{Crit}$ and the result follows. □

Lemma 2.7 *Let f be a C^1 map of the interval I with a finite number of critical points. For any subset $J \subset I$ one has $HD(\bigcup_{n\geq 0} f^{-n}(J)) = HD(J)$.*

Proof: It suffices to show that $\mathrm{HD}(f^{-n}(J)) \leq \mathrm{HD}(J)$ for each n. But f^n is C^1 and has a finite number of critical points and sends $f^{-n}(J)$ into J. Apply Lemma 2.6 to conclude. $\square$

Definition 2.8 *The* forward orbit $O^+(x)$ *of a point x is defined by*

$$O^+(x) = \{f^n(x) : n \geq 0\}$$

and that of a set S by

$$O^+(S) = \bigcup_{x\in S} O^+(x).$$

Definition 2.9 *Call a critical point c* primary *if $c \notin \overline{O^+(f(Crit))}$, and* primary with respect to a set K *if $c \notin \overline{O^+(f(Crit \cap K))}$.*

Lemma 2.10 *Given a continuous map f, if $y \in \omega(x)$ then $\omega(y) \subset \omega(x)$.*

Proof: Follows from the definition and continuity. $\square$

Corollary 2.11 *For a continuous map f, if $y \in \omega(x)$ and $z \in \omega(y)$ then $z \in \omega(x)$.*

Corollary 2.12 *For a continuous map f, if $y \in \overline{O^+(x)}$ and $z \in \overline{O^+(y)}$ then $z \in \overline{O^+(x)}$.*

Proof: Note that $\omega(x) = \bigcap_{n>0} \overline{O^+(f^n(x))}$. $\square$

Lemma 2.13 *Suppose there is a finite, strictly positive number of critical points in a set K, and that all these critical points are non-recurrent. Then there exists a critical point which is primary with respect to K.*

Proof: Consider all sequences $(c_1c_2\ldots)$ such that $c_i \in \mathrm{Crit}\cap K$ and $c_{i+1} \in \overline{O^+(f(c_i))}$. We claim that there is a sequence of maximal length (possibly equal to 1). For otherwise there would be a sequence of length greater than #Crit, and two occurrences of the same critical c in the same sequence, together with Corollary 2.12, imply c is not non-recurrent, contradiction.

Now consider a maximal such sequence $(c_1 \ldots c_r)$. We claim c_1 is primary with respect to K. For if $c_1 \in \overline{O^+(f(c_0))}$ say for some $c_0 \in K\cap$ Crit, then $(c_0c_1 \ldots c_r)$ is a longer such sequence, contradicting maximality. $\square$

3. Dimension Estimate for Induced Maps

Let $A \subset B$ be measurable subsets of an interval endowed with the usual subspace topology. Let t be a real number and $g : A \to B$ a C^1 function.

Definition 3.1 *A measure m on B is called* t-conformal *if*

$$m(g(C)) = \int_C |Dg(x)|^t dm, \tag{3}$$

whenever C is measurable, $g(C)$ is measurable and $g_{|C}$ is injective. It is also a probability measure *if $m(B) = 1$.*

Let $\{J_i\}_{i=1}^{\infty}$ be a countable collection of pairwise-disjoint, open subintervals of an open interval U satisfying:

- $\overline{J_i} \cap \overline{J_j} = \emptyset$ for all $i \neq j$;
- $\overline{J_i} \cap \partial U = \emptyset$ for all i.

Let $\phi : \bigcup_{i\geq 1} J_i \to U$ be a C^1 (or C^2) function such that for each i, the restriction $\phi_{|J_i}$ is a diffeomorphism between J_i and U. Suppose also that ϕ is expanding, i.e. there exists λ such that $|D\phi| \geq \lambda > 1$, and that on each branch $\sup_{x\in J_i} |D\phi| < M_i$ for some constant $M_i < \infty$.

Let $Z_n = \bigcup_{i=1}^{n} J_i$, $Z = \bigcup_{n=1}^{\infty} Z_n$, and set

$$\begin{aligned} L_n &= \{x \in Z_n : \phi^j(x) \in Z_n \ \forall j \geq 0\} \\ &= \bigcap_{j=0}^{\infty} \phi^{-j}(Z_n) \\ &= \bigcap_{j=0}^{\infty} \phi^{-j}(Z_n \cap \phi^{-1}(\overline{Z_n})) \end{aligned}$$

and $L = \{x \in Z : \phi^j(x) \in Z \ \forall j \geq 0\}$.

Let ϕ_n be the restriction of ϕ to Z_n. Then L_n is completely invariant with respect to ϕ_n and is compact, as a countable intersection of closed sets, thus it is hyperbolic. It is long-known [5] that there exists an atom-free t_n-conformal probability measure m_n for $\phi_n : Z_n \to U$ with support on L_n, where t_n is the Hausdorff dimension of L_n.

Lemma 3.2 *There exists a t-conformal probability measure m for $\phi : Z \to U$ where*

$$t = \lim_{n\to\infty} HD(L_n).$$

Proof: Let m be a weak limit point of the t_n-conformal probability measures m_n for $\phi_n : Z_n \to U$. We first show t-conformality of m for $\phi : J_i \to I$ for each i. This is a standard convergence argument.

Suppose $A \subset J_i$ satisfies $m(\partial A) = m(\partial\phi(A)) = 0$. Then, using conformality of m_n for $\phi : J_i \to I$ when $n > i$,

$$m(\phi(A)) = \lim_{n\to\infty} m_n(\phi(A)) = \lim_{n\to\infty} \int_A |D\phi|^{t_n} dm_n = \int_A |D\phi|^t dm.$$

Now we deal with subsets A compactly contained in J_i. For $\varepsilon > 0$ consider an open cover of A by a countable family of sets $\{B_j\}$ satisfying $m(\partial B_j) = m(\partial\phi(B_j)) = 0$ and $B_j \subset J_i$ and

$$m(\bigcup_j B_j \setminus A) < \frac{\varepsilon}{M_i}.$$

We can now define inductively a partition $\{C_j\}$ of $\bigcup_j B_j$ as follows:

$$C_1 = B_1, C_{j+1} = B_{j+1} \setminus \bigcup_{k=1}^{j} B_j.$$

We have shown m is t-conformal for each C_j. Thus

$$\begin{aligned} m(\phi(A)) &\leq m(\bigcup_j \phi(C_j)) = \sum_j m(\phi(C_j)) \\ &= \sum_j \int_{C_j} |D\phi|^t dm = \int_{\bigcup_j C_j} |D\phi|^t dm \\ &= \int_A |D\phi|^t dm + \int_{\bigcup_j C_j} |D\phi|^t dm \leq \int_A |D\phi|^t dm + \varepsilon. \end{aligned}$$

This is true for all $\varepsilon > 0$ so $m(\phi(A)) \leq \int_A |D\phi|^t dm$. In the other direction, using the same partition,

$$\begin{aligned} m(\phi(A)) =& \sum_j (m(\phi(C_j)) - m(\phi(C_j \setminus A))) \\ \geq& \sum_j \int_{C_j} |D\phi|^t dm - \sum_j \int_{C_j \setminus A} |D\phi|^t dm \\ \geq& \int_A |D\phi|^t dm - \varepsilon. \end{aligned}$$

If A is open and $\partial A \cap \partial J_i \neq \emptyset$, let $\overline{C_j} \subset A$ be increasing subsets of A such that $\bigcup_{j\geq 1} C_j = A$.

$$\lim_{j\to\infty} m(\phi(A \setminus C_j)) = \lim_{j\to\infty} m(A \setminus C_j) = 0$$

implies m is t-conformal for A. We conclude m is t-conformal for $\phi : J_i \to I$ for all i.

Each open set containing points of L contains an interval which is sent by some iterate of ϕ onto I, and thus is of positive measure, if I is. We show that for any J_i compactly contained in I, $m(\overline{J_i}) \geq \frac{1}{M_i}$:

$$\begin{aligned} m_n(\phi(J_i)) &= m_n(I) = 1 \\ &= \int_{J_i} |D\phi|^{t_n} dm_n \\ &\leq M_i^{t_n} m_n(J_i) \\ &\leq M_i m_n(J_i) \\ m_n(J_i) \quad &\geq \frac{1}{M_i}. \end{aligned}$$

Letting $n \to \infty$, $m(\overline{J_i}) \geq \frac{1}{M_i}$, as required, so m is not trivial (zero) on U.

Defining $m' = \frac{1}{m(I)} m$ we obtain a t-conformal probability measure as required. $\square$

Proposition 3.3 *Suppose that there is a constant C such that for all n and all x, y belonging to a connected component of the domain of ϕ^n,*

$$\frac{D\phi^n(x)}{D\phi^n(y)} \leq C.$$

Then the Hausdorff dimension of L is equal to $t = \lim_{n\to\infty} HD(L_n)$.

Proof: Of course $L \supset L_n$ so $\mathrm{HD}(L) \geq t$. Let m be the measure from the preceding lemma. Consider a pairwise-disjoint, countable cover of L by $\{C_j\}_{j\geq 1}$ where for each j there exists $n_j > 0$ such that $\phi^{n_j} : C_j \to U$ is a diffeomorphism. For every $\varepsilon > 0$ there exists such a cover with $\mathrm{diam}(C_j) < \varepsilon$ for all j since $|D\phi^{-1}| \leq \lambda^{-1} < 1$. From above, the distortion of ϕ^{n_j} on each C_j is bounded by some constant C independent of j and ε.

We find a finite bound for $\sum_{j\geq 1}\mathrm{diam}(C_j)^t$ independently of ε which shows $\mathrm{HD}(L) \leq t$:

$$\begin{aligned}
m(U) &= \int_{C_j} |D\phi^{n_j}(x)|^t dm \\
&\leq C^t |D\phi^{n_j}(x_0)|^t m(C_j). \\
\mathrm{diam}(U)^t &= \left(\int_{C_j} |D\phi^{n_j}(x)| d\lambda\right)^t \\
&\geq C^{-t} |D\phi^{n_j}(x_0)|^t \mathrm{diam}(C_j)^t
\end{aligned}$$

for some fixed $x_0 \in C_j$. Dividing, we find

$$\frac{\mathrm{diam}(C_j)^t}{m(C_j)} \leq \frac{C^{2t}\mathrm{diam}(U)^t}{m(U)}$$

which gives

$$\mathrm{diam}(C_j)^t \leq \frac{C^{2t}\mathrm{diam}(U)^t}{m(U)} \cdot m(C_j).$$

Since m is a probability measure and the C_j are pairwise disjoint the sum of $m(C_j) \leq 1$ and

$$\sum_{j\geq 1} \mathrm{diam}(C_j)^t \leq \frac{C^{2t}\mathrm{diam}(U)^t}{m(U)} < \infty,$$

as required. □

We shall now apply these results to induced maps. Let K be a forward invariant compact set for a D-map f.

Definition 3.4 *We say (ϕ, Z, K) is an* induced Markov system *if there is an open interval U such that for all $W \in \tau(K)$, one has $U \cap O^+(\partial W) = \emptyset$ and there is a collection of intervals $\{J_i\}_{i=1}^{\infty}$ with $Z = \bigcup_i J_i$ and a map $\phi : Z \to U$ such that the following hold:*

- *$K \cap J_i \neq \emptyset$ for all i;*
- *on each J_i there is an $n_i > 0$ such that $\phi_{|J_i} = f^{n_i}_{|J_i}$;*
- *$\overline{J_i} \cap \overline{J_j} = \emptyset$ for all $i \neq j$;*
- *$\overline{J_i} \cap \partial U = \emptyset$ for all i;*
- *$\phi : J_i \to U$ is a diffeomorphism onto U;*

- *there is a constant C such that for all $n > 0$ and all x, y belonging to a same connected component of the domain of ϕ^n,*
$$\frac{D\phi^n(x)}{D\phi^n(y)} \leq C.$$

As before, let $Z_n = \bigcup_{i=1}^{n} J_i$, $Z = \bigcup_{n=1}^{\infty} Z_n$, and set

$$\begin{aligned} L_n \quad &= \{x \in Z_n : \phi^j(x) \in Z_n \ \forall j \geq 0\} \\ \text{and } L &= \{x \in Z : \phi^j(x) \in Z \ \forall j \geq 0\}. \end{aligned}$$

We can then apply Proposition 3.3 to get

$$\mathrm{HD}(L) = \lim_{n\to\infty} \mathrm{HD}(L_n). \tag{4}$$

This will be interesting if we show that $L \subset K$ and that L_n is contained in a hyperbolic set.

Lemma 3.5 *Let (ϕ, Z, K) be an induced Markov system for a D-map f and let $L = \{x \in Z : \phi^j(x) \in Z \ \forall j \geq 0\}$. Then $L \subset K$.*

Proof: By definition, $K \cap Z$ is non-empty. Suppose $x \in Z$ satisfies $\phi(x) \in K$. Let i be such that $x \in J_i$. Let n_i be such that $\phi_{|J_i} = f^{n_i}_{|J_i}$. By definition, for all $W \in \tau(K)$ the forward orbit of ∂W under f is disjoint from U. Thus for all $0 \leq n < n_i$, we have $f^n(J_i) \cap \partial W = \emptyset$. Again by definition, $J_i \cap K \neq \emptyset$, so for all $0 \leq n < n_i$, we have $f^n(J_i) \cap W = \emptyset$ and thus $x \in K$. Therefore $K \cap Z$ is backward-invariant with respect to ϕ. The set of all inverse images of x by ϕ is contained in K and is dense in L. By compactness of K, $L \subset K$. □

Lemma 3.6 *For all $n > 0$, the set L_n considered above is contained in a hyperbolic set for f.*

Proof: The set

$$\bigcup_{i=1}^{n} \bigcup_{j=0}^{n_i} f^j(J_i \cap L_n)$$

is a forward invariant compact and is at a positive distance from all critical and parabolic points. By the Mañé Hyperbolicty Theorem [6] it is a hyperbolic subset of K containing L_n. □

Proposition 3.7 *Let (ϕ, Z, K) be an induced Markov system for a D-map f and let $L = \{x \in Z : \phi^j(x) \in Z \ \forall j \geq 0\}$. Then HD(L) $\leq$ HypD(K).*

Proof: Each L_n is contained in a hyperbolic set so $\mathrm{HD}(L_n) \leq \mathrm{HypD}(K)$. Apply equation (4) to conclude. □

4. Parabolic Estimates

Let K be a compact subset of the interval and let f be K-equivalent to a function of class $M_0(K)$. Our goal in this section is to show that the set of points in K whose forward orbits accumulate on the parabolic periodic points has Hausdorff dimension less than the hyperbolic dimension of K.

Definition 4.1 *For a D-map f and forward-invariant set K, denote by Rec(x, K) the set of points $y \in K$ such that $x \in \omega(y)$. If S is a set then*

$$\mathit{Rec}(S, K) = \bigcup_{x \in S} \mathit{Rec}(x, K).$$

Proposition 4.2 *Let K be a compact set and f a map of class $M_0(K)$ with a parabolic periodic point p such that $f(p) = p$ and $Df(p) = 1$. Then*

$$\mathit{HD}(\mathit{Rec}(p, K)) \leq \mathit{HypD}(K).$$

Proof: We shall construct an expanding induced map and apply the estimates of the Section 3.

Recall that $\tau(K)$ is finite and for all $W \in \tau(K), \partial W$ is preperiodic. Thus the set of points

$$S = \bigcup_{n \geq 0} \bigcup_{W \in \tau(K)} f^n(\partial W)$$

is finite. Since f is of class $M_0(K)$, all critical points are mapped directly to the attracting boundary ∂I.

Let U be a small, regularly returning, open interval whose closure is disjoint from $S \cup \partial I \cup \mathrm{Crit}$ and such that $p \in \partial U$. If there is a fixed point q in U then $\mathrm{HD}((p, q)) = 0$. Henceforth, suppose there is no fixed point in U and that $f(U) \setminus U \neq \emptyset$, so p is topologically repelling in U.

Now U is compactly contained I and all branches of the first return map to U, or indeed any subinterval of I, extend to diffeomorphisms onto the interior of I. By Fact 2.2, there exists a constant $C > 1$ such that the first return map to any regularly returning interval contained in U has distortion bounded by C. Indeed, one can take

$$C^{-1} = \frac{e^{-\sigma(|I|)|I|}}{(1 + \nu(U, I))^2}.$$

Denote by g the branch of f^{-1} which fixes p. Since p is a parabolic point,

$$\lim_{n \to \infty} \frac{|g^n(U) \setminus g^{n+1}(U)|}{|g^{n+1}(U)|} = 0.$$

Thus there exists $N > 0$ such that if $n \geq N$ then

$$\frac{|g^n(U) \setminus g^{n+1}(U)|}{|g^{n+1}(U)|} \leq \frac{1}{2C^2}.$$

Let V be the largest of the intervals $g^n(U) \setminus g^{n+1}(U)$ where $n \geq N$, realised for $n = n_0$, and let T be the interval $g^{n_0+1}(U)$. Then $\frac{|V|}{|T|} < \frac{1}{2C^2}$. Denote by ψ_V the first entry map to V and by ψ_T the first entry map to T. Both of these maps have distortion bounded by C.

Consider the map $\phi = \psi_T \circ \psi_{V|T}$, where $\psi_{V|T}$ is the restriction of ψ_V to T. The domain of ϕ is contained in T, and ϕ has distortion bounded by C^2. Let J be a connected component of the domain of ϕ. Then $J \subset g^n(U) \setminus g^{n+1}(U)$ for some $n > n_0$, so

$$|J| \leq |V| \leq \frac{1}{2C^2}|T|.$$

J is mapped by ϕ onto T, so there exists $x \in J$ such that $|D\phi(x)| \geq 2C^2$. For all $y \in J$ one deduces

$$|D\phi(y)| \geq \frac{1}{C^2}2C^2 = 2,$$

so $|D\phi| \geq 2$ everywhere on its domain of definition. By Lemma 2.5, there is a bound C' on the distortion of branches of ϕ^n independent of the $n \geq 1$.

Denote by Z the union of the connected components of the domain of ϕ which have non-empty intersection with K. Then (ϕ, Z, K) is an induced Markov system. Let

$$L = \{x \in K \cap T : \phi^n(x) \in Z \text{ for all } n \geq 0\}.$$

By Proposition 3.7, $\mathrm{HD}(L) \leq \mathrm{HypD}(K)$. Note that L contains all points in $\mathrm{Rec}(p, K) \cap T$ whose forward orbits accumulate on p in T.

The same argument for the other side of the parabolic point p combined with Lemma 2.7 allows us to conclude that $\mathrm{HD}(\mathrm{Rec}(p, K)) < \mathrm{HypD}(K)$. □

Theorem 4 *Let K be a compact subset of the interval and let f be K-equivalent to a function of class $M_0(K)$. Then*

$$HD(K) = HypD(K).$$

Proof: By Lemma 1.7 we can assume f is of class $M_0(K)$. Iterates f^n of f are then of class $M_0(K)$. Hyperbolic dimensions with respect to f and with respect to iterates f^n coincide. Let p be a parabolic point of period n say. Then $p, f(p), \ldots, f^{n-1}$ are all fixed points of the map f^{2n} with $Df^{2n}(f^i(p)) = 1$. Denote by $\pi(K)$ the parabolic points contained in a set K. We apply Proposition 4.2 for the map f^{2n} for each $n > 0$ to get that

$$\mathrm{HD}\left(\bigcup_{p\in\pi(K)} \mathrm{Rec}(p, K)\right) \leq \mathrm{HypD}(K).$$

Consider $S = K \setminus (\bigcup_{p\in\pi(K)} \mathrm{Rec}(p, K))$. It remains to show that $\mathrm{HD}(S) \leq \mathrm{HypD}(K)$. For $\delta > 0$ let B_δ be the δ-neighbourhood of the parabolic set $\pi(K)$, i.e. the set of points at a distance $< \delta$ from $\pi(K)$. Put

$$K_\delta = K \setminus \bigcup_{i\geq 0} f^{-i}(B_\delta).$$

One can then write

$$S = \bigcup_{\delta>0} K_\delta,$$

so

$$\mathrm{HD}(S) = \sup_{\delta>0} \mathrm{HD}(K_\delta).$$

Each K_δ is hyperbolic by the Mañé Hyperbolicity Theorem and every hyperbolic set is contained in a K_δ for some $\delta > 0$. $\square$

5. Critical Estimates

Let K be a compact subset of the interval and let f be K-equivalent to a function of class $M_n(K)$. Our goal in this section is to show that the set of points whose forward orbits accumulate on the critical set has Hausdorff dimension less than the hyperbolic dimension of K. Recall that for a point x, $\mathrm{Rec}(x, K)$ was defined in Definition 4.1.

Proposition 5.1 *Let K be a τ-set for a D-map f with a primary critical point c. Then*

$$\mathit{HD}(\mathit{Rec}(c, K)) \leq \mathit{HypD}(K).$$

Proof: As per the previous section, we shall construct an expanding induced map and apply the results of Section 3.

Since c is primary, there exists an open, regularly returning neighbourhood V' of c such that every branch of the first return map to V' is a diffeomorphism onto V'. Similarly, for all regularly returning intervals $U \subset V'$, every branch of the first return map to U is a diffeomorphism onto U which extends to a diffeomorphism onto V'. Of course, the domain of the first return map to U does not contain c.

If $\mathrm{HD}(\mathrm{Rec}(c, K)) \neq 0$ then branches of the first return map to V' accumulate on c. By Fact 2.2, for all $C_0 > 1$ there exists $\delta > 0$ such that if V is a regularly returning, open neighbourhood of c and $|V| < \delta$ then the first return map has distortion bounded by C_0. Fix C_0 with $1 < C_0 < \frac{4}{3}$ and a corresponding δ.

An elementary argument gives that C^2 non-flatness of the critical points implies that f is almost symmetric near each critical point. That is, if f_-^{-1} is the branch of f^{-1} sending $f(c)$ to c such that $f_-^{-1}(x) \leq c$ for all x in its domain, and $\varepsilon > 0$, then

$$\lim_{\varepsilon\to 0} \left| \frac{c - f_-^{-1}(f(c+\varepsilon))}{\varepsilon} \right| = 1.$$

Then there exists a $\delta' > 0$ such that if V is a neighbourhood of c, $f(\partial V)$ is a point (such a V is sometimes called symmetric) and $|V| < \delta'$, then for all intervals $W \subset V$ which do not contain c, $\frac{|W|}{|V|} < \frac{2}{3}$.

Consider such a V which is regularly returning and also satisfies $|V| < \delta$. Let W be a connected component of the domain of the first return map ϕ_V to V. Then $\frac{|W|}{|V|} < \frac{2}{3}$ and ϕ_V sends W onto V so there is a point $x \in W$ such that $|D\phi_V(x)| > \frac{3}{2}$. The distortion bound then implies that $|D\phi_V| > \frac{9}{8}$ on W and thus everywhere on its domain.

Since ϕ_V satisfies $|D\phi_V| \geq \frac{9}{8} > 1$ we can apply Lemma 2.5 so there exists a bound C on the distortion of branches of ϕ_V^n independent of $n \geq 1$.

Recall that $\tau(K)$ is finite and for all $W \in \tau(K), \partial W$ is preperiodic. Thus the set of points

$$S = \bigcup_{n \geq 0} \bigcup_{W \in \tau(K)} f^n(\partial W)$$

is finite.

Let $U \subset V$ be a small, regularly returning, open interval disjoint from S and such that $c \in \partial U$. Also assume that ∂U is not in the domain of ϕ_V, so that branches of the first return map ϕ_U to U accumulate on ∂U.

Denote by Z the union of the connected components of the domain of ϕ_U which have non-empty intersection with K. Then (ϕ_U, Z, K) is an induced Markov system. Let

$$L = \{x \in K \cap T : \phi_U^n(x) \in Z \text{ for all } n \geq 0\}.$$

By Proposition 3.7, $\mathrm{HD}(L) \leq \mathrm{HypD}(K)$. Note that L contains all points in $\mathrm{Rec}(c, K) \cap U$ whose forward orbits accumulate on p in U.

The same argument for the other side of the critical point c combined with Lemma 2.7 allows us to conclude that $\mathrm{HD}(\mathrm{Rec}(c, K)) < \mathrm{HypD}(K)$. □

Theorem 5 *For all integers $n \geq 0$ the following statement holds. Let K be a compact subset of the interval and let f be K-equivalent to a function of class $M_n(K)$, then*

$$HD(Rec(Crit, K)) \leq HypD(K).$$

Proof: We shall provide a proof by induction. Note that the statement of the theorem is trivially true if $n = 0$. Suppose that the statement holds for all $n \leq N - 1$. We must show that it holds for $n = N$. Consider a compact set K and map f such that f is K-equivalent to a function of class $M_N(K)$. By Lemma 1.7 we can assume f is of class $M_N(K)$. By Lemma 2.13 f has a primary critical point c. Now apply Proposition 5.1 to get

$$\mathrm{HD}(\mathrm{Rec}(c, K)) \leq \mathrm{HypD}(K).$$

Let $S = \mathrm{Rec}(\mathrm{Crit}, K) \setminus \mathrm{Rec}(c, K)$. It thus remains to show that $\mathrm{HD}(S) \leq \mathrm{HypD}(K)$. For $\delta > 0$ let B_δ be the ball of radius δ around the primary critical point c and let

$$K_\delta = K \setminus \bigcup_{i \geq 0} f^{-i}(B_\delta).$$

One can then write

$$S = \bigcup_{\delta > 0} \mathrm{Rec}(\mathrm{Crit}, K_\delta) \cup \bigcup_{i \geq 0} f^{-i}(c),$$

so

$$\mathrm{HD}(S) = \sup_{\delta > 0} \mathrm{HD}(\mathrm{Rec}(\mathrm{Crit}, K_\delta)).$$

Since c is primary, for sufficiently small δ the ball B_δ is disjoint from $O^+(f(\mathrm{Crit}))$. Thus f is K_δ-equivalent to a map of class $M_{N-1}(K_\delta)$. Applying the inductive hypothesis gives

$$\mathrm{HD}(\mathrm{Rec}(\mathrm{Crit}, K_\delta)) \leq \mathrm{HypD}(K_\delta) \leq \mathrm{HypD}(K).$$

Therefore $\mathrm{HD}(S) \leq \mathrm{HypD}(K)$ as required. □

6. Proof of Theorem 3

Given a compact subset K of an interval, let f be K-equivalent to a map of class $M_n(K)$. We wish to show that the Hausdorff and hyperbolic dimensions of K coincide. By Lemma 1.7 we can assume f is of class $M_n(K)$. We need the following lemmas:

Lemma 6.1 *Let U be a connected component of $I \setminus K$ and let $y \in \partial U \cap K$. Then y is preperiodic.*

Proof: If $f^i(U) \in \tau(K)$ for some $i \geq 0$ then $f^i(y)$ is preperiodic, so y is and we are done. Henceforth we assume $f^i(U) \cap K = \emptyset$ for all $i \geq 0$, so $f^i(U) \notin \tau(K)$ for any i.

Any critical points in $I \setminus K$ are mapped directly into the attracting boundary since f is of class $M_n(K)$. By the absence of wandering intervals ([1], theorem A, p 267), either some iterate of U contains a critical point of f (in $I \setminus K$) or has non-empty intersection with an immediate basin of attraction of a periodic attractor ([1], lemma II.3.1). In either case, we get that there exists an $m > 0$ and a $p > 0$ such that $f^m(U)$ has non-empty intersection with the immediate basin of a periodic attractor A of period p say. Now $f^m(y) \in K$, so it is not contained in any basin of attraction. The boundary ∂A is periodic of period p or $2p$. If $f^m(y) \in \partial A$ we are done; otherwise $f^m(U)$ contains a point x of ∂A. This point acts as an anchor: $f^{m+2kp}(U) \ni x$ for all $k \geq 0$. If k is not preperiodic then $f^{m+p}(U) \cup f^{m+2p}(U) \ni f^m(k)$, contradicting $f^i(U) \cap K = \emptyset$ for all $i \geq 0$. □

Lemma 6.2 *Preperiodic points are dense in the Julia set of f.*

Proof: This actually hold for all maps of class D. Lebesgue almost every point either is contained in the basin of attraction of a periodic point or accumulates on a critical point. The boundary of the basins of attraction consists of preperiodic points (periodic points at the boundary of the immediate basins). Thus preperiodic points are dense anywhere the Julia set is not connected.

Now let U be an interval contained in the Julia set. It suffices to show that $\overline{U}$ contains a preperiodic point. By Fact 2.4, the size of $f^i(U)$ is larger than some constant ϵ for all $i \geq 0$. Thus $X := \overline{\bigcup_{i\geq 0} f^i(U)}$ is a finite union of intervals which get mapped into each other by f, so at least one of them contains a periodic point. We claim that there exists an $m > 0$ such that $\bigcup_{i=0}^{m} f^i(\overline{U}) = X$.

Indeed, suppose otherwise. Since f is continuous and has only a finite number of critical points, there is an $n > 0$ such that $X \setminus \bigcup_{i=0}^{n} f^i(\overline{U})$ is a finite collection of intervals mapped monotonically into each other by f. But then they cannot be contained in the Julia set, which is a contradiction since the Julia set is completely invariant set which thus contains all iterates of U. □

Lemma 6.3 *For each $\delta > 0$ there exists a neighbourhood V_δ of $Crit \cap K$, which satisfies the following conditions:*

- *each connected component of V_δ contains at least one critical point;*

- *every point $x \in V_\delta \cap K$ satisfies dist$(x, Crit \cap K) \leq \delta$;*
- *the boundary of V_δ is made up entirely of preperiodic points contained in $K \cap \partial I$.*

Proof: Let c be in Crit$\cap K$. Let J be a connected component of $I \setminus \{c\}$. If $K \cap J \cap B(c, \delta) = \emptyset$, let p be the point in $J \cap (K \cup \partial I)$ which is closest to c, so $(p, c) \cap K = \emptyset$. By Lemma 6.1 p is preperiodic. If $K \cap J \cap B(c, \delta) \neq \emptyset$ then $J \cap B(c, \delta)$ either contains connected components of $I \setminus K$ or (non-exclusively) contains an interval contained in K. Either way, there is a preperiodic point p contained in $K \cap J \cap B(c, \delta)$. One finds in the same way a point p' for the other side of c.

Define V_δ as the union of intervals (p, p') constructed in this manner around each critical point in Crit$\cap K$. It satisfies the claims of the lemma. □

We now conclude the proof of Theorem 3. Let $S = K \backslash \mathrm{Rec}(\mathrm{Crit}, K)$. By Theorem 5 it only remains to show that $\mathrm{HD}(S) \leq \mathrm{HypD}(K)$.

$$K_\delta = K \setminus \bigcup_{i \geq 0} f^{-i}(V_\delta),$$

where V_δ is given by the preceding lemma for δ small. Due to the properties of V_δ, K_δ is a τ-set: we have only added at most a finite number of sets to $\tau(K)$, each being a connected component of V_δ with preperiodic boundary. One can then write

$$S = \bigcup_{\delta > 0} K_\delta \cup \bigcup_{i \geq 0} f^{-i}(\mathrm{Crit} \cap K),$$

so

$$\mathrm{HD}(S) = \sup_{\delta > 0} \mathrm{HD}(K_\delta).$$

It now suffices to show that $\mathrm{HD}(K_\delta) = \mathrm{HypD}(K_\delta)$, since $K_\delta \subset K$. But f is K_δ-equivalent to a map of class $M_0(K_\delta)$. Applying Theorem 4 gives the required result.

7. Lebesgue 0, Hausdorff 1

Recall the cadre of Theorem 2:

We suppose for D-map f that there exists a closed interval U with the following properties:

- there exists a flat parabolic point of period q such that $p \in \partial U$;
- U contains no critical points of f^{2q};
- $\frac{f^{2q}(x)-p}{x-p} > 1$ for all $x \in U$;
- there exists $V = [y, f^{2q}(y)]$ contained in U and an integer n such that $f^n(V) \supset U$.

Consider the set $W = U \cup f(V) \cup \cdots \cup f^{n-1}(V)$. The collection of points K that stay in W forever form a non-empty, forward-invariant, compact set. We can then study hyperbolic subsets of W. The Denker-Urbanski construction (similarly to Section 3) on increasing hyperbolic subsets of K gives in the limit a t-conformal probability measure m on K with $t = \mathrm{HypD}(K) \leq 1$. We shall show $t = 1$.

For simplicity, let $F = f^{2q}$, let g be the branch of F^{-1} which fixes p and suppose, without loss of generality, that $p \leq x$ for all $x \in U$.

Since m is conformal and $f^{2n}(V)$ covers K it follows that $m(V) > 0$. One has $\sum_{n\geq 0} |g^n(V)| \leq |U|$ so by Fact 2.2 there is a constant C such that

$$\frac{DF^n(x)}{DF^n(y)} < C$$

for all $x, y \in g^n(V)$ for all $n > 0$. Thus by t-conformality (as per Section 3)

$$|g^n(V)|^t \leq \frac{C|V|^t}{m(V)} m(g^n(v)).$$

Summing both sides

$$\sum_{n\geq 0} |g^n(V)|^t \leq \frac{C|V|^t}{m(V)} < \infty. \tag{5}$$

Up as far as here is well-known (see [10]). Let

$$b(x) = \sup\{g^n(y) : g^n(y) < x\}.$$

Then $b(x)$ is constant on the interior of each interval $g^n(V)$ and

$$|g^n(V)|^t = |F(g^n(y)) - g^n(y)|^t = \int_{g^n(V)} |F(b(x)) - b(x)|^{t-1}.$$

Summing one gets

$$\sum_{n\geq 0} |g^n(V)|^t = \int_p^{F(y)} |F(b(x)) - b(x)|^{t-1} < \infty,$$

where the finiteness of the sum and integral comes from inequality (5).

Now for each positive number r, by flatness of the parabolic point, there exists a $\delta > 0$ such that for all $x \in (p, p+\delta]$ one has $x < F(x) < x + (x-p)^r$. Also $b(x) < x$, thus

$$|F(b(x)) - b(x)| < (x-p)^r.$$

Since $t - 1 \leq 0$ we have

$$\int_p^{F(y)} |F(b(x)) - b(x)|^{t-1} \geq \int_p^{p+\delta} (x-p)^{r(t-1)}$$

which is integrable only if $r(t-1) > -1$. But this means $t > \frac{r-1}{r}$ and this holds for all positive numbers r so $t \geq 1$. Thus $t = 1$ and Theorem 2 is proven. $\square$

To finish, we show that such maps can exist. Let $a > 0$ and on the interval $[-a, 1+\frac{a}{3}]$ let the function f satisfy:

- $f(-a) = -a$;
- $f(x) < x$ for all x such that $-a < x < 0$;
- if $x \in [0, \frac{1}{3}]$ then $f(x) = x + e^{3-\frac{1}{x}+\log\frac{2}{3}}$;
- if $x \in [\frac{2}{3}, 1+\frac{a}{3}]$ then $f(x) = 3 - 3x$.
- on $[-a, 1+\frac{a}{3}]$ the function f is C^2 and has only one critical point, necessarily contained in $[\frac{1}{3}, \frac{2}{3}]$, and this critical point is non-flat.

Then f is a D-map with a flat parabolic point at 0 which satisfies Theorem 2 - one can take $U = [0, \frac{1}{3}]$. Then $f(U) = f([\frac{2}{3}, 1]) = [0, 1]$. The critical point is in the basin of attraction of the fixed point at a, and the Julia set for f is the complement of this basin of attraction. It is well-known and easy to show that the Julia set is of Lebesgue measure 0, for example by showing that density points cannot exist.

We have constructed a D-map satisfying Corollary 1.5, that is, a map whose Julia set has hyperbolic (and therefore Hausdorff) dimension equal to 1 but is of Lebesgue measure 0.

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