The Glimm space of the minimal tensor product of C*-algebras

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Let $A$ be a C*-algebra and $\text{Prim}(A)$ the set of kernels of irreducible representations of $A$.

- Hull-kernel topology on $\text{Prim}(A)$ has open sets $\mathcal{U}(I) = \{ P \in \text{Prim}(A) : P \nsubseteq I \}$ for $I$ an ideal of $A$,
- The map $P \mapsto P \cap I$ is a homeomorphism $\mathcal{U}(I) \to \text{Prim}(I)$,
- $I = \bigcap\{ P \in \text{Prim}(A) : P \supseteq I \}$,
- In particular $\bigcap\{ P : P \in \text{Prim}(A) \} = \{0\}$.
- $\text{Prim}(A)$ is a locally compact $T_0$ space, but non-Hausdorff in general.
Representation over $\text{Prim}(A)$

Let $A$ be a commutative $C^*$-algebra.

- Irreducible representations of $A$ are the characters on $A$,
- Hull-kernel topology is precisely the weak-$*$ topology,
- Gelfand-Naimark: $A = C_0(\text{Prim}(A))$.

Non-commutative generalisation:

- Try to represent $A$ as the section algebra of a bundle over $\text{Prim}(A)$, with fibres $\{A/P : P \in \text{Prim}(A)\}$,
- Identify $a \in A$ with the section $\hat{a} : \text{Prim}(A) \to \bigsqcup A/P$, where $\hat{a}(P) = a + P$,
- J.M.G Fell: $A$ is $*$-isomorphic to the section algebra of a bundle over $\text{Prim}(A)$ iff $\text{Prim}(A)$ hull-kernel Hausdorff.
- J. Dauns and K.H. Hofmann obtained a sectional representation theorem valid for any $C^*$-algebra $A$ over the complete regularisation of $\text{Prim}(A)$.
Glimm(A)

- Define an equivalence relation $\approx$ on $\text{Prim}(A)$ via $P \approx Q$ if and only if $f(P) = f(Q)$ for all $f \in C^b(\text{Prim}(A))$,

- As a set let $\text{Glimm}(A) = \text{Prim}(A)/\approx$, and $\rho_A : \text{Prim}(A) \to \text{Glimm}(A)$ be the quotient map,

- For $f \in C^b(\text{Prim}(A))$ define $f^\rho : \text{Glimm}(A) \to \mathbb{C}$ via $f^\rho([P]) = f(P)$,

- Topology $\tau_{cr}$ on $\text{Glimm}(A)$ induced by $\{f^\rho : f \in C^b(\text{Prim}(A))\}$,

- $\text{Glimm}(A)$ is completely regular (Tychonoff), and the map $f \mapsto f^\rho$ is a *-isomorphism of $C^b(\text{Prim}(A))$ onto $C^b(\text{Glimm}(A))$,

- To each $\approx$-equivalence class $p = [P]$ define the closed two-sided ideal $G_p = \bigcap[P]$, the Glimm ideal of $A$ corresponding to $p$.

- Note that if $\text{Prim}(A)$ is Hausdorff then $\text{Prim}(A) = \text{Glimm}(A)$.
The Dauns-Hofmann Theorem

- Dauns and Hofmann: can represent $A$ as the section algebra of a $C^*$-bundle over $\text{Glimm}(A)$, with fibres $\{A/G_p : p \in \text{Glimm}(A)\}$,
- $\text{ZM}(A) \equiv C^b(\text{Glimm}(A)) \equiv C(\beta\text{Glimm}(A))$.
- For each $p \in \text{Glimm}(A)$ we have $G_p = M_p \cdot A$, where $M_p = \{f \in C^b(\text{Glimm}(A)) : f(p) = 0\}$.

Let $X$ be a locally compact Hausdorff space,

- G.G. Kasparov: $A$ can be represented as the section algebra of an upper-semicontinuous $C^*$-bundle over $X$ if and only if there is a $*$-homomorphism $\mu : C_0(X) \to \text{ZM}(A)$ such that $\mu(C_0(X)) \cdot A = A$,
- Equivalently can take $\mu' : C_0(X) \to C^b(\text{Glimm}(A))$ with this property,
- This occurs if and only if there is a continuous map $\phi : \text{Prim}(A) \to X$, or equivalently, a continuous map $\phi^p : \text{Glimm}(A) \to X$. 

The Glimm space of the minimal tensor product of $C^*$-algebras
Glimm($A \otimes B$)

For C*-algebras $A$ and $B$, let $A \otimes B$ be their minimal C*-tensor product. In order to address these questions for $A \otimes B$, need to determine Glimm($A \otimes B$) in terms of Glimm($A$) and Glimm($B$):

1. as a topological space,
2. as a collection of ideals of $A \otimes B$.

Related questions studied in:

The minimal tensor product

Let $A$ and $B$ be $C^*$-algebras.

- Denote by $A \circledcirc B$ the $*$-algebraic tensor product of $A$ and $B$, and by $A \otimes B$ their minimal $C^*$-tensor product,

- For $*$-homomorphisms $\pi : A \to C$ and $\sigma : B \to D$, let $\pi \circledcirc \sigma : A \circledcirc B \to C \circledcirc D$ be the $*$-homomorphism defined via

$$ (\pi \circledcirc \sigma)(a \circledcirc b) = \pi(a) \otimes \sigma(b) $$

for elementary tensors $a \circledcirc b \in A \circledcirc B$, and $\pi \otimes \sigma : A \otimes B \to C \otimes D$ the $*$-homomorphism extending $\pi \circledcirc \sigma$.

- Recall that if $\pi$ and $\sigma$ are injective (resp. surjective) then so is $\pi \otimes \sigma$. 

The Glimm space of the minimal tensor product of $C^*$-algebras
Ideals of $A \otimes B$, Property (F)

For any C*-algebra $C$ let $\text{Id}'(C)$ be the set of all proper norm-closed 2-sided ideals of $C$.

- If $(I, J) \in \text{Id}'(A) \times \text{Id}'(B)$ and $q_I : A \to A/I$, $q_J : B \to B/J$ the quotient maps then $q_I \circ q_J : A \circ B \to (A/I) \circ (B/J)$ has
  \[ \ker(q_I \circ q_J) = I \circ B + A \circ J, \]
  which has closure $I \otimes B + A \otimes J$ in $A \otimes B$ by injectivity.

- Alternatively, we may consider the kernel of $q_I \otimes q_J : A \otimes B \to (A/I) \otimes (B/J)$,

- Clearly $\ker(q_I \otimes q_J) \supseteq I \otimes B + A \otimes J$, but this inclusion may be strict,

- If $\ker(q_I \otimes q_J) = I \otimes B + A \otimes J$ for all $(I, J) \in \text{Id}'(A) \times \text{Id}'(B)$ we say that $A \otimes B$ satisfies Tomiyama’s property (F).

- S. Wassermann: $B(H) \otimes B(H)$ does not satisfy (F).
Maps between $\text{Id}'(A) \times \text{Id}'(B)$ and $\text{Id}'(A \otimes B)$

We define maps $\Phi, \Delta' : \text{Id}'(A) \times \text{Id}'(B) \to \text{Id}'(A \otimes B)$ via

$$\Phi(I, J) = \ker(q_I \otimes q_J)$$
$$\Delta'(I, J) = I \otimes B + A \otimes J$$

- The restriction of $\Phi$ to $\text{Prim}(A) \times \text{Prim}(B)$ is a homeomorphism onto its image, which is a dense subset of $\text{Prim}(A \otimes B)$ (Wulfsohn),
- For $I \triangleleft A \otimes B$, define ideals of $A$ and $B$ via
  $$I^A = \{a \in A : a \otimes B \subseteq I\}, \quad I^B = \{b \in B : A \otimes b \subseteq I\},$$
- Setting $\Psi(I) = (I^A, I^B)$ gives a map $\Psi : \text{Id}'(A \otimes B) \to \text{Id}'(A) \times \text{Id}'(B)$,
- For $(I, J) \in \text{Id}'(A) \times \text{Id}'(B)$,
  $$\quad (\Psi \circ \Phi)(I, J) = (\Psi \circ \Delta')(I, J) = (I, J)$$
The complete regularisation of $\text{Prim}(A) \times \text{Prim}(B)$

- Consider the relation $\approx$ on $\text{Prim}(A) \times \text{Prim}(B)$,
- Easily seen that $(P_1, Q_1) \approx (P_2, Q_2)$ if and only if $P_1 \approx P_2$ and $Q_1 \approx Q_2$,
- Hence as a set, the complete regularisation of $\text{Prim}(A) \times \text{Prim}(B)$ agrees with $\text{Glimm}(A) \times \text{Glimm}(B)$,
- Topology $\tau_{cr}$ induced by $C^b(\text{Prim}(A) \times \text{Prim}(B))$ is stronger than the product topology $\tau_p$. 

The Glimm space of the minimal tensor product of $C^*$-algebras
Characterisation of Glimm($A \otimes B$)

Theorem

Let $A$ and $B$ be $C^*$-algebras and $\rho_A$, $\rho_B$ and $\rho_\alpha$ the complete regularisation maps of Prim($A$), Prim($B$) and Prim($A \otimes B$) respectively. Denote by $\Delta$ the restriction of $\Delta'$ to Glimm($A$) $\times$ Glimm($B$). Then

(i) $\Delta$ is a homeomorphism of (Glimm($A$) $\times$ Glimm($B$), $\tau_{cr}$) onto Glimm($A \otimes B$),

(ii) $\Delta$ is an open bijection of (Glimm($A$) $\times$ Glimm($B$), $\tau_p$) onto Glimm($A \otimes B$),

(iii) For all $(P, Q) \in$ Prim($A$) $\times$ Prim($B$) we have

$$(\Delta \circ (\rho_A \times \rho_B))(P, Q) = (\rho_\alpha \circ \Phi)(P, Q),$$

(iv) $\Delta^{-1} = \Psi$.

Extends a result of E. Kaniuth by eliminating the assumption of (F).
\( \tau_{cr} \) and \( \tau_p \)

**Theorem**

Let \( A \) be a \( C^* \)-algebra satisfying one of the following conditions:

(i) \( \text{Prim}(A) \) is compact (e.g. \( A \) unital),
(ii) \( A \) is \( \sigma \)-unital and \( \text{Glimm}(A) \) is locally compact,
(iii) The complete regularisation map \( \rho_A \) is open (e.g. \( \text{Prim}(A) \) Hausdorff, \( A \) quasi-standard).

Then \( \tau_{cr} = \tau_p \) on \( \text{Glimm}(A) \times \text{Glimm}(B) \) for any \( C^* \)-algebra \( B \), so that \( \text{Glimm}(A \otimes B) \) can be identified with \( (\text{Glimm}(A) \times \text{Glimm}(B), \tau_p) \).

On the other hand, there is a separable, liminal \( C^* \)-algebra \( A \) such that \( \tau_{cr} \neq \tau_p \) on \( \text{Glimm}(A) \times \text{Glimm}(A) \). Hence \( \text{Glimm}(A \otimes A) \) is not homeomorphic to this space with the product topology.
Sectional representation

Consider the fibred space \( \mathcal{A} = \{A/G_p : p \in \text{Glimm}(A)\} \):

- Each element \( a \in A \) defines a section \( \hat{a} : \text{Glimm}(A) \to \mathcal{A} \) such that \( \hat{a}(p) = a + G_p \),

- \( \|a\| = \sup \{\|\hat{a}(p)\| : p \in \text{Glimm}(A)\} \),

- norm functions \( p \mapsto \|\hat{a}(p)\| \) upper semicontinuous on \( \text{Glimm}(A) \) for all \( a \in A \).

- Dauns-Hofmann: there is a topology on \( \mathcal{A} \) such that
  1. the restriction of this topology to each fibre \( A/G_p \) is the norm topology, and
  2. the collection \( \{\hat{a} : a \in A\} \) consists of all continuous sections \( \text{Glimm}(A) \to \mathcal{A} \),

- \( p \mapsto \|\hat{a}(p)\| \) continuous for all \( a \in A \) iff \( \rho_A \) is an open map (Lee). In this case we say that \( A \) defines a \textit{continuous} \( C^* \)-bundle over \( \text{Glimm}(A) \). Equivalently, \( A \) is a continuous \( C_0(\text{Glimm}(A)) \)-algebra.
Sectional representation of $A \otimes B$

As a consequence of the previous two Theorems, we can represent $A \otimes B$ as the section algebra of an upper semicontinuous bundle s.t.:

- The base space is given by $(\text{Glimm}(A) \times \text{Glimm}(B), \tau_{cr})$,
- Fibre algebras given by
  \[
  \left\{ \frac{A \otimes B}{\Delta(G_p, G_q)} : (p, q) \in \text{Glimm}(A) \times \text{Glimm}(B) \right\}
  \]
- If in fact $\Delta(G_p, G_q) = \Phi(G_p, G_q)$ for all $(p, q) \in \text{Glimm}(A) \times \text{Glimm}(B)$, then the fibre algebras are given by $(A/G_p) \otimes (B/G_q)$.
- This assumption is strictly weaker than property (F).
Continuous C*-bundles

Corollary

Suppose that $A$ and $B$ are C*-algebras such that $\Phi = \Delta$ on $\text{Glimm}(A) \times \text{Glimm}(B)$, and denote by $\rho_A, \rho_B$ and $\rho_\alpha$ the complete regularisation maps of $\text{Prim}(A), \text{Prim}(B)$ and $\text{Prim}(A \otimes B)$ respectively. Then the following are equivalent:

(i) The Dauns-Hofmann representations of $A$ and $B$ define continuous C*-bundles over $\text{Glimm}(A)$ and $\text{Glimm}(B)$ respectively,

(ii) $\rho_A$ and $\rho_B$ are open maps,

(iii) $\rho_\alpha$ is an open map,

(iv) The Dauns-Hofmann representation of $A \otimes B$ defines a continuous C*-bundle over $\text{Glimm}(A \otimes B)$.
Embedding $ZM(A) \otimes ZM(B)$ in $ZM(A \otimes B)$

- We may regard $M(A) \otimes M(B) \subseteq M(A \otimes B)$, with action on $A \otimes B$ determined via

$$(x \otimes y)(a \otimes b) = (xa) \otimes (yb), \quad (a \otimes b)(x \otimes y) = (ax) \otimes (by)$$

for elementary tensors $a \otimes b \in A \otimes B$ and $x \otimes y \in M(A) \otimes M(B)$,

- Akemann, Pedersen and Tomiyama: If $A$ is $\sigma$-unital and non-unital, and $B$ infinite dimensional, then this inclusion is strict.

We will describe the centres of both algebras in terms of $\text{Glimm}(A)$ and $\text{Glimm}(B)$, and give necessary and sufficient conditions for equality.

- R. Haydon and S. Wassermann: for any $C^*$-algebras $C$ and $D$, $Z(C \otimes D) = Z(C) \otimes Z(D)$,

- In particular, we have $Z(M(A) \otimes M(B)) = ZM(A) \otimes ZM(B)$,

- Moreover, easily seen that $ZM(A) \otimes ZM(B) \subseteq ZM(A \otimes B)$. 

The Glimm space of the minimal tensor product of $C^*$-algebras
The structure space of the centre

Corollary to Dauns-Hofmann Theorem:

\[ ZM(A) = C^b(\text{Prim}(A)) = C^b(\text{Glimm}(A)) = C(\beta\text{Glimm}(A)) \]

It follows that

\[ ZM(A \otimes B) = C(\beta\text{Glimm}(A \otimes B)) = C(\beta (\text{Glimm}(A) \times \text{Glimm}(B), \tau_{cr})) \]

While

\[ ZM(A) \otimes ZM(B) = C(\beta\text{Glimm}(A)) \otimes C(\beta\text{Glimm}(B)) = C(\beta\text{Glimm}(A) \times \beta\text{Glimm}(B)) \]
Conditions for \(ZM(A) \otimes ZM(B) = ZM(A \otimes B)\)

Theorem
\[ZM(A) \otimes ZM(B) = ZM(A \otimes B) \text{ if and only if either}\]
1. One of \(\text{Glimm}(A)\) or \(\text{Glimm}(B)\) is finite, or
2. \(\tau_{cr} = \tau_p\) on \(\text{Glimm}(A) \times \text{Glimm}(B)\), and \(\text{Glimm}(A) \times \text{Glimm}(B)\) is pseudocompact.

(Recall that a completely regular space \(X\) is said to be pseudocompact if every \(f \in C(X)\) is bounded.)
The embedding \(ZM(A) \otimes ZM(B) \hookrightarrow ZM(A \otimes B)\) is dual to the continuous surjection

\[\tilde{\iota} : \beta (\text{Glimm}(A) \times \text{Glimm}(B), \tau_{cr}) \to \beta \text{Glimm}(A) \times \beta \text{Glimm}(B),\]

extending the identity \(\iota\) on \(\text{Glimm}(A) \times \text{Glimm}(B)\). Hence equality holds if and only \(\tilde{\iota}\) is a homeomorphism. Note that \(\tau_{cr} = \tau_p\) is a necessary condition.
Stone-Čech compactification of a product space

- Suppose that $\text{Glimm}(B)$ is finite, hence discrete and compact,
- Then $\rho_B : \text{Prim}(B) \to \text{Glimm}(B)$ is an open map, so that $\tau_{cr} = \tau_p$,
- $\text{Glimm}(A) \times \text{Glimm}(B)$ consists of $n$ disjoint copies of $\text{Glimm}(A)$,
- Hence

$$\beta(\text{Glimm}(A) \times \text{Glimm}(B)) = \beta\text{Glimm}(A) \times \text{Glimm}(B) = \beta\text{Glimm}(A) \times \beta\text{Glimm}(B).$$

In the infinite case, we apply Glicksberg’s Theorem: for completely regular spaces $X$ and $Y$, the canonical map $\beta(X \times Y) \to \beta X \times \beta Y$ is a homeomorphism if and only if $X \times Y$ is pseudocompact.
A non-trivial example $\text{ZM}(A) \otimes \text{ZM}(B) = \text{ZM}(A \otimes B)$

Let $X$ be a non-compact locally compact Hausdorff space, and $H$ a separable infinite dimensional Hilbert space.

- For $x \in \beta X$, $Y := \beta X \setminus \{x\}$ is locally compact, non-compact and pseudocompact,
- Set $A = C_0(Y, B(H))$ (with pointwise operations and supremum norm), then $\text{Glimm}(A) = Y$,
- Define $B$ as the $C^*$-algebra of sequences $(T_n) \subset B(H)$, such that $T_n \to T_\infty \in K(H)$,
- $\text{Glimm}(B) = \mathbb{N} \cup \{\infty\}$, and $\rho_B : \text{Prim}(B) \to \text{Glimm}(B)$ is open (Archbold and Somerset),
- Since $B$ is $\sigma$-unital and non-unital, $M(A) \otimes M(B) \subset M(A \otimes B)$ (Akemann, Pedersen and Tomiyama),
- $\text{Glimm}(A) \times \text{Glimm}(B)$ is the product of a pseudocompact space and a compact space, hence is pseudocompact, and $\tau_{cr} = \tau_p$,
- It follows that $\text{ZM}(A) \otimes \text{ZM}(B) = \text{ZM}(A \otimes B)$. 

The Glimm space of the minimal tensor product of $C^*$-algebras
Example cont’d.

- $A \otimes B$ has a quotient isomorphic to $B(H) \otimes B(H)$, hence does not have property (F),

- On the other hand, $\Phi = \Delta$ on $\text{Glimm}(A) \times \text{Glimm}(B)$ (Archbold), and $\tau_{cr} = \tau_{p}$,

- Hence $A \otimes B$ is $\ast$-isomorphic to the section algebra of a bundle over $Y \times \mathbb{N} \cup \{\infty\}$ (with the product topology), with fibre algebras for $y \in Y$ given by

  $$(\mathcal{A} \otimes \mathcal{B})(y,n) = B(H) \otimes B(H), \ n \in \mathbb{N}$$
  $$(\mathcal{A} \otimes \mathcal{B})(y,\infty) = B(H) \otimes K(H)$$

- Since $A$ and $B$ define continuous $C^\ast$-bundles and $\Delta = \Phi$, so does $A \otimes B$. 