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Theorem: $(l^\infty, ||\cdot||_\infty) = (BC(N,K), ||\cdot||_\infty)$ is a Banach space (i.e. is complete).
Proof: Consider a Cauchy sequence $(x_n)_{n=1}^\infty$ in l^∞. Each x_n is thus an element of l^∞ so that

$$x_n = (x_{n_j})_{j=1}^\infty = (x_{n_1}, x_{n_2}, \ldots)$$

$$||x_n||_\infty = \sup_j |x_{nj}|$$

$(x_n)_{n=1}^\infty$ Cauchy means: Given any $\varepsilon > 0$ there exists some natural number N so that

$$\begin{aligned} n, m \geq N \quad &\Rightarrow \quad ||x_n - x_m||_\infty < \varepsilon \\ &\Rightarrow \quad ||(x_{n_j} - x_{m_j})_{j=1}^\infty||_\infty < \varepsilon \\ &\Rightarrow \quad \sup_j |x_{n_j} - x_{m_j}| < \varepsilon \end{aligned}$$

$\therefore$ If we fix any j we have:
Given $\varepsilon > 0$ $\exists$ N s.t.

$$n, m \geq N \Rightarrow |x_{n_j} - x_{m_j}| < \varepsilon$$

$\therefore$ $(x_{n_j})_{j=1}^\infty$ is a Cauchy sequence in K
$\therefore$ $\exists$ $y_j = \lim_{n\to\infty} x_{n_j} \in K$ (Since K is complete)

$$\begin{array}{rllll} x_1 = & (x_{11}, & x_{12}, & x_{13}\ldots, x_{1j}, \ldots) \\ x_2 = & (x_{21}, & x_{22}, & x_{23}\ldots, x_{2j}, \ldots) \\ x_3 = & (x_{31}, & x_{32}, & x_{33}\ldots, x_{3j}, \ldots) \\ & \downarrow & \downarrow & \quad\quad\downarrow \\ & y_1 & y_2 & \quad\quad y_j \end{array}$$

Put $y = (y_j)_{j=1}^\infty = (y_1, y_2, y_3 \ldots)$ We need two things to show l^∞ is complete:
(a) $y \in l^\infty$
(b) $\lim_{n\to\infty} x_n = y$ in l^∞ <u>or</u> $\lim_{n\to\infty} ||y - x_n||_\infty = 0$

Proof of (a): To show $y \in l^\infty$ we use the Cauchy criterion with $\varepsilon = 1$ to find $N = N_1$ so that:

$$n, m \geq N_1 \Rightarrow ||x_n - x_m||_\infty < 1$$

Take $m = N = N_1$. Then:

$$n \geq N_1 \Rightarrow ||x_n - x_{N_1}||_\infty < \varepsilon = 1$$

Or in other words:

$$\sup_j |x_{n_j} - x_{n_1 j}| < 1$$

For each j we have:

$$\begin{aligned} n \geq N_1 \Rightarrow |x_{n_j} - x_{N_1 j}| \quad &< \quad 1 \\ \Rightarrow |x_{n_j}| \quad &\leq \quad |x_{n_j} - x_{N_1 j}| + |x_{N_1 j}| \\ &\leq \quad 1 + |x_{N_1 j}| \\ &\leq \quad 1 + ||x_{N_1}||_\infty \\ \Rightarrow |y_j| = \lim_{n\to\infty} |x_{n_j}| \quad &\leq \quad 1 + ||x_{N_1}||_\infty \\ \Rightarrow \sup_j |y_j| \quad &\leq \quad 1 + ||x_{N_1}||_\infty < \infty \end{aligned}$$

$\therefore$ We have shown that $y \in l^\infty$
Proof of (b):
To show finally that $\lim_{n\to\infty} ||x_n - y||_\infty = 0$ take $\varepsilon > 0$. Using the Cauchy condition we can find $N = N_\varepsilon > 0$ so that:

$$n, m \geq N \Rightarrow ||x_n - x_m||_\infty < \frac{\varepsilon}{2}$$

For any fixed $j \geq 1$ and $n \geq N$ (also fixed for now) we have:

$$m \geq N \Rightarrow |x_{n_j} - x_{m_j}| \leq ||x_n - x_m||_\infty < \frac{\varepsilon}{2}$$

$$\therefore \lim_{m\to\infty} |x_{n_j} - x_{m[j}| = |x_{n_j} - y_j| \leq \frac{\varepsilon}{2}$$

$$\therefore n \geq N \Rightarrow ||x_n - y||_\infty = \sup_j |x_{n_j} - y_j| \leq \frac{\varepsilon}{2} < \varepsilon$$

We've proved $\lim_{n\to\infty} ||x_n - y||_\infty = 0$

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Theorem: $(BC(X, \mathbb{K}), ||\cdot||_\infty)$ is a Banach space for any metric (or topological) space X.

Proof: This is quite similar to the above proof for l^∞, but we have to take care of continuity.
Let $(f_n)_{n=1}^\infty$ be a Cauchy sequence in $BC(X, \mathbb{K})$. Fix any point $t \in X$. We claim that the sequence of scalars $(f_n(t))_{n=1}^\infty$ is Cauchy (i.e pointwise Cauchy).
Given $\varepsilon > 0 \exists\ N > 0$ so that:

$$\begin{aligned} n, m \geq N \quad &\Rightarrow \quad ||f_n - f_m||_\infty < \varepsilon \\ &\Rightarrow \quad \sup_{x\in X} |f_n(x) - f_m(x)| < \varepsilon \end{aligned}$$

Take just $x = t$ we get:

$$n, m \geq N \Rightarrow |f_n(t) - f_m(t)| < \varepsilon$$

$\therefore$ $(f_n(t))_{n=1}^\infty$ is Cauchy in $\mathbb{K}$ $\therefore$ $\lim_{n\to\infty}$ exists in $\mathbb{K}$ We can define a function $g : X \to \mathbb{K}$ by $g(t) = \lim_{n\to\infty} f_n(t)$ (This function g is now the 'pointwise limit' of the functions $(f_n)_{n=1}^\infty$)
We need to know more:

(a) g is bounded
(b) g is continuous
(c) $\lim_{n\to\infty} ||f_n - g||_\infty = 0$ so that $\lim_{n\to\infty} f_n = g$ in $(BC(X, \mathbb{K}), ||\cdot||_\infty)$

(a) Fix $\varepsilon = 1$ and choose $N > 0$ so that

$$n, m \geq N \Rightarrow ||f_n - f_m||_\infty < 1$$

Then take $m = N, n \geq N$. We have

$$n \geq N \Rightarrow ||f_n - f_m||_\infty < 1$$

For any $t \in X$ we have:

$$n \geq N \Rightarrow |f_n(t) - f_N(t)| \leq ||f_n - f_N||_\infty < 1$$

$$\Rightarrow |f_n(t)| \leq |f_n(t) - f_N(t)| + |f_N(t)| < 1 + ||f_N||_\infty$$

$$\therefore\ |g(t)| = \lim_{n\to\infty} |f_n(t)| \leq 1 + ||f_N||_\infty$$

True $\forall\ t \in X$.

$$\therefore\ \sup_{t\in X} |g(t)| \leq 1 + ||f_n||_\infty < \infty$$

i.e. g is a bounded function.
(b) To show g is continuous, we use the fact that$f_n \to g$ uniformly on X as $n \to \infty$ together with the fact that uniform limits of continuous functions are continuous.

`My notes seem to end here`

Beastiality isn't half as common in the West of Ireland as I make it out to be [1]

[1] Michael Alfonsus Gallagher