BPS Bound States and Quivers

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Based on joint work with B. Pioline and A. Sen.
BPS Bound States in 4 Dimensions ⇐⇒ Quiver Mathematics

\[ \gamma_{13} \]

\[ \mathbb{C}^{N_1} \]

\[ \gamma_{12} \]

\[ \mathbb{C} \]

\[ \gamma_{23} \]

\[ \mathbb{C}^{N_2} \]

\[ \mathbb{C}^{N_3} \]
Outline

1. BPS (bound) states
2. Relation to moduli spaces of quiver representations
3. The Coulomb branch formula
\( \mathcal{N} = 2 \) supersymmetry

Anti-commuting extension of the Poincaré algebra:

\[
\begin{align*}
\{ Q^I_\alpha, \bar{Q}^J_\beta \} &= 2 \sigma^\mu_{\alpha\beta} P_\mu \delta^{IJ}, & I, J &= 1, 2, \\
\{ Q^I_\alpha, Q^J_\beta \} &= 2 \varepsilon_{\alpha\beta} \varepsilon^{IJ} Z(\gamma, t),
\end{align*}
\]

Central charge \( Z(\gamma, t) \): \((\Gamma, \mathcal{M}) \rightarrow \mathbb{C}\).

With \( \mathcal{M} \) the moduli space of vacua.
BPS states

States in 1-particle Hilbert space $\mathcal{H}(t)$ are parametrized by:
- momenta $P_\mu$
- spin $j$
- discrete (electric-magnetic) charges $\gamma \in \Gamma \cong \mathbb{Z}^{2r}$

BPS states:
- are invariant under half of the susy generators
- their mass satisfies the BPS bound: $M(\gamma, t) = P_0 = |Z(\gamma, t)|$
Supersymmetric index

\[ \Omega(\gamma; t) = -\frac{1}{2} \text{Tr}_{\mathcal{H}(\gamma,t)} (2J_3)^2 (-1)^{2J_3} \]

with \( J_3 \) generator of \( SU(2)_{\text{spin}} \)

- Measure for the number of quantum states
- Projects to the Hilbert space of BPS states \( \mathcal{H}_{\text{BPS}}(\gamma, t) \)
- Protected quantity, independent of coupling constant and continuous parameters (hypermultiplet scalars)
- Closely related to mathematics of Calabi-Yau manifolds via geometric engineering using string theories
Wall-crossing

Dependence of $\Omega(\gamma; t)$ on vector multiplet scalars $t$

A central charge $Z(\gamma, t) \in \mathbb{C}$ is associated to every BPS state

- If $Z(\gamma_1, t) \parallel Z(\gamma_2, t)$: $M(\gamma_1, t) + M(\gamma_2, t) = M(\gamma_1 + \gamma_2, t)$
  $\Rightarrow$ bound states are marginally stable

- $\Omega(\gamma_1 + \gamma_2; t)$ is only locally constant as function of $t$;
  it might jump across walls where $Z(\gamma_1, t) \parallel Z(\gamma_2, t)$. 
Wall-crossing

The jump $\Delta \Omega(\gamma; t^+ \to t^-) = \Omega(\gamma; t^-) - \Omega(\gamma; t^+)$ is given by the wall-crossing formula's

Denef, Moore (2007); Kontsevich, Soibelmann (2008); Joyce, Song (2008); ...  

KS wall-crossing formula:

- Lie algebra $\mathcal{A}$: $[e_{\gamma_1}, e_{\gamma_2}] = (-1)^{\langle \gamma_1, \gamma_2 \rangle} \langle \gamma_1, \gamma_2 \rangle e_{\gamma_1+\gamma_2}$
- Lie group element: $U_{\gamma}(t) = \exp \left( \Omega(\gamma; t) \sum_{n \geq 1} \frac{e_n \gamma}{n^2} \right)$
- Invariance of infinite product:

$$\prod_{\gamma \in \Gamma, Z(\gamma, t^+) \in V} U_{\gamma}(t^+) = \prod_{\gamma \in \Gamma, Z(\gamma, t^-) \in V} U_{\gamma}(t^-)$$

Puts strong constraints on the BPS indices $\Omega(\gamma; t)$
BPS bound states of $\mathcal{N} = 2$ supergravity

$N$ BPS black holes with charges $\gamma_i$ located at $\vec{r}_i$ in $\mathbb{R}^3$

Static BPS bound states exist due to interplay between gravitational attraction and electro-magnetic repulsion

$\implies$ Bound states are static and therefore part of the 1-particle Hilbert space $\mathcal{H}_{\text{BPS}}(\gamma; t)$
Denef equations

\[ \mathcal{N} = 2 \text{ BPS equations} \text{ of motion require the distances} \]
\[ r_{ij} = |\vec{r}_i - \vec{r}_j| \in \mathbb{R}_+ \text{ to satisfy:} \]
\[ \sum_{j=1}^{N} \frac{\gamma_{ij}}{r_{ij}} = c_i(\{\gamma_k\}; t) \]

- \( \gamma_{ij} = \langle \gamma_i, \gamma_j \rangle \in \mathbb{Z} \): Dirac-Schwinger-Zwanziger innerproduct
- \( c_i(\{\gamma_j\}; t) \in \mathbb{R} \): stability parameters depending on \( Z(\gamma_i, t) \)

Denef (2000)

Phase space \( M_N(\{\gamma_i\}, \{c_i\}) \):

- parametrizes \( \vec{r}_i \in \mathbb{R}^3, i = 1, \ldots, N \)
- has dimension \( 2N - 2 \)

De Boer, El-Showk, Messamah, Van den Bleeken (2008)
Denef equations: Two aspects

Wall-crossing: Solutions might decay or recombine upon varying $c_i \in \mathbb{R}$:

Denef (2000); Denef, Moore (2007),

For example $N = 2$: \( \lim_{c_1 \to 0} r_{12} = \lim_{c_1 \to 0} \frac{\gamma_{12}}{c_1} = \pm \infty \)

Scaling solutions: Centers could get arbitrarily close, depending on \( \{\gamma_i\} \)

Bena, Wang, Warner (2006); Denef, Moore (2007),

For example $N = 3$: If \( \gamma_{12} + \gamma_{23} \geq \gamma_{31} \), and cyclic perm. \( \Rightarrow \)

\( \lim_{\lambda \to 0} r_{ij}(\lambda) = \lambda \gamma_{ij} + \mathcal{O}(\lambda^2) \in M(\{\gamma_i\}, \{c_i\}) \)
Decomposition formula

\[ \tilde{\Omega}(\gamma; t) = \sum_{\sum_i N_i \gamma_i = \gamma, \gamma_i \neq \gamma_j, i \neq j} g(\{N_i \gamma_i\}; \{c_i(t)\}) \prod_j \frac{\tilde{\Omega}_S(\gamma_j)^{N_j}}{N_j!} \]

with

- \( \Omega_S(\gamma) = \text{single-centered index:} \)
  
  \# of states associated to the single center BH with charge \( \gamma \).

  Independent of \( t \) and expected to enumerate \( j = 0 \) states

  Sen (2009)

- \( \tilde{\Omega}_S(\gamma_j) = \sum_n |\gamma_j| \frac{\Omega_S(\gamma_j/n)}{n^2} \): rational invariant associated to center \( j \)


- \( \frac{\tilde{\Omega}_S(\gamma)^N}{N!} \): Maxwell-Boltzmann distribution

- \( g(\{N_i \gamma_i\}; \{c_i\}) \in \mathbb{Z}: \) \# of “binding” states

  JM, Pioline, Sen (2010)

**Question:** What is the meaning of \( g(\{N_i \gamma_i\}; \{c_i\}) \)?
Quiver quantum mechanics: Field content

Low energy excitations of the black hole bound state

$\mathcal{N} = 4$ quiver quantum mechanics

Denef (2002); Denef, Moore (2007); ... Familiar from BPS monopoles: Cederwall, Ferretti, Nilsson, Salomonson (1995); Sethi, Stern, Zaslow (1995); Gauntlett, Harvey (1996); Gauntlett, Kim, Park, Yi (2000),...

Lagrangian is determined by:

- multiplicities $\{N_i\}$ and innerproducts $\{\gamma_{ij}\}$:
  - vector multiplets $(\vec{r}_i, A_i, \lambda_i)$ with gauge group $U(N_i)$
  - $|\gamma_{ij}|$ bifundamental chiral multiplets
    $(\phi^a_{ij}, F^a_{ij}, \psi^a_{ij}), a = 1, \ldots, |\gamma_{ij}|$

- Fayet-Iliopoulos parameters $\vec{c}$

- parametrized by a quiver $(\vec{N}, \vec{c})$: 

\[ U(N_1), c_1 \xrightarrow{\gamma_{13}} U(N_3), c_3 \]

\[ \xrightarrow{\gamma_{12}} U(N_2), c_2 \]
**Quiver quantum mechanics: Higgs branch**

**Higgs branch**

- vector multiplets are integrated out

\[
\sum_{j \rightarrow i}^{\gamma_{ij}} \phi_{ij}^a (\phi_{ij}^a)^\dagger - \sum_{j \rightarrow i}^{\gamma_{ji}} (\phi_{ij}^a)^\dagger \phi_{ij}^a = c_i \mathbf{1}_{N_i}
\]

\[
\frac{\partial W(\{\phi_{ij}^a\})}{\partial \phi_{kl}^{b}} = 0
\]

⇒ equations for symplectic quotient

\[
\mathcal{M}(\vec{N}, \vec{c}) = \{\text{equations}\} // \prod_{i} U(N_i)
\]

Algebraic quotients more manageable for explicit analysis
Quiver

1. Vertices $i \in V$
2. Arrows $i \rightarrow j \in A$

Quiver representation:
A set of vector spaces $S_i \cong \mathbb{C}^{N_i}$ and $\forall i \rightarrow j$ a linear map $S_i \rightarrow S_j$

Stable representation:
Given $\vec{c} \in \mathbb{R}^{|V|}$, a rep. $F$ is stable if $\forall F' \subset F$, $\frac{\vec{c} \cdot \vec{N'}}{|N'|} < \sum_i \frac{\vec{c} \cdot \vec{N}}{N_i}$

Moduli space $\mathcal{M}(\vec{N}; \vec{c})$ of stable representations modulo
$\prod_i GL(N_i, \mathbb{C})$
Topological invariants of moduli spaces

- **Poincaré polynomial** \( p(X, y) = \sum_{i=0}^{2\dim_{\mathbb{C}} X} b_i(X) y^i \)
  
  No oriented loops: Reineke’s formula determines \( p(\mathcal{M}(\vec{N}, \vec{c}), y) \) Reineke (2002)

- **\( \chi_y \)-genus** \( \chi_y(X) = \sum_{p, q=0}^{\dim_{\mathbb{C}} X} h^{p, q}(X)(-1)^p y^q \)
  
  Index computations (Jeffrey-Kirwan residues) in principle determine \( \mathcal{M}(\vec{N}, \vec{c}) \)
  

- **Euler number** \( \chi(X) = p(X, -1) = \chi_{-1}(X) \)
Meaning of $g(\{N_i \gamma_i; \{c_i\}\})$

Euler number of moduli space of Abelian quiver representations:

$$g(\{N_i \gamma_i; \{c_i\}\}) = \chi(\vec{1}_N, \vec{c}_N)$$

with

$$\vec{1}_N = (1, \ldots, 1, 1, \ldots, 1, \ldots, 1)$$

with

$$\vec{c}_N = (c_1, \ldots, c_1, c_2, \ldots, c_2, \ldots, c_{|V|})$$

Consequence of Maxwell-Boltzmann distribution
Corollary: Abelianization formula

**Specialization of decomposition formula:**

\[ \Omega_S(\ell \gamma_i) = \delta_{\ell,1}, \quad i \in \{\text{nodes}\} \]

Then \( \bar{\Omega}(\gamma; t) \) corresponds to moduli space of **non-Abelian** representations

\[ \Rightarrow \text{non-Abelian from Abelian:} \]

\[ \chi(\mathcal{M}_Q(\vec{N}; \vec{c})) \sim \sum_{Q'} \chi(\mathcal{M}_{Q'}(\vec{1}_N'; \vec{1}_c')) \]

JM, Pioline, Sen (2010)

**Example:**

\[ U(2) \rightarrow U(1), 2\gamma_1 \rightarrow U(1), 2\gamma_2 \]

Mathematical studies by Mozgovoy, Okada, Reineke, Stoppa, Weist,...

Related to GW/Kronecker correspondence
Can we compute the $\Omega(\gamma; t)$ using the perspective of bound states?

Consider the Coulomb branch $|c_i| \to 0$
- chiral multiplets are integrated out
- vector multiplets: $\sum_{j=1, j \neq i}^{N} \frac{\gamma_{ij}}{r_{ij}} = c_i(\{\gamma_k\}; t)$

Denef (2002); Kim, Park, Wang, Yi (2011),

N.B.: BPS solutions require more conditions to be physical, in particular regularity of metric
Goal: determine $g(\{\gamma_i\}; \{c_i\})$ in space-time

1. $g(\{\gamma_i\}, \{c_i\})$ is the (twisted) Dirac index of the space $M_N(\{\gamma_i\}, \{c_i\})$

2. $\exists$ symplectic form on $M_N$:

$$\omega = \frac{1}{2} \sum_{i<j} \gamma_{ij} \epsilon^{abc} \frac{dr^a_{ij} \wedge dr^b_{ij} r^c_{ij}}{|r_{ij}|^3}$$

3. $g(\{\gamma_i\}, \{c_i\})$ can be determined using geometric quantization in special cases

De Boer, El-Showk, Messamah, Van den Bleeken (2008); ...
Coulomb branch

General computation is feasible by refining the index:

\[ \Omega(\gamma, y; t) = \frac{\text{Tr}_{\mathcal{H}_{BPS}(\gamma, t)} (-y)^{2J_3}}{-y^{-2} + 2 - y^2} \]

- Poincaré polynomial of \( \mathcal{M}(\gamma, t) \)
- \( g(\{\gamma_i\}, y; \{c_i\}): \text{equivariant Dirac index of } M_N(\{\gamma_i\}, \{c_i\}) \)

Index theorem:

\[ g(\{\gamma_i\}, y; \{c_i\}) = \int_{M_N} \text{Ch}(\mathcal{L}, \nu) \hat{A}(M_N, \nu)|_{2N-2} \]

with \( \nu = \log(y) \), \( \text{Ch}(\mathcal{L}, \nu) = \text{equivariant Chern character of } \mathcal{L} \), \( \hat{A}(X, \nu) = \text{equivariant } \hat{A}-\text{genus of } X \)

Berline, Vergne (1985)
Coulomb branch: Localization

Evaluate integral by localization with respect to $J_3$

Duistermaat, Heckman (1982); Berline, Vergne (1985);…

⇓

Sum over isolated fixed points $\in M_N(\{\gamma_i\}, \{c_i\})$ of $J_3$

The solutions which contribute are of the form:

\[\gamma_1 \gamma_3 \gamma_2\]

z-axis

JM, Pioline, Sen (2011)
Coulomb branch formula

Fixed point formula:

\[ g(\{\gamma_i\}, y; \{c_i\}) = \frac{(-1)^{\sum_{{i < j}} \gamma_{ij} + N - 1}}{(y - y^{-1})^{N-1}} \sum_{p \in \{\text{f.p. of } J_3\}} s(p) y^{2J_3(p)} \]

- angular momentum:

\[ J_3(p) = \frac{1}{2} \sum_{i < j} \gamma_{ij} \text{sign}(z_j - z_i) \]

- sign:

\[ s(p) = \text{sign} \left( \det \left( \frac{\partial^2 W}{\partial z_i \partial z_j} \right) \right) \]

with \( W(\{z_i\}) = - \sum_{i < j} \gamma_{ij} \text{sign}(z_j - z_i) \log |z_i - z_j| - \sum_{i=1}^{N} c_i z_i \)
Example: \( \gamma_i, i = 1, \ldots, 3 \), such that \( \gamma_{12}, \gamma_{13}, \gamma_{23} > 0 \), \( c_3 < c_2 < 0 < c_1 \)

- **Fixed points have orderings:**
  \( \{1, 2, 3; +\} \), \( \{2, 1, 3; −\} \), \( \{3, 1, 2; −\} \), \( \{3, 2, 1; +\} \),

- **Enumerate:**

\[
g(\{\gamma_i\}, y; \{c_i\}) = (-1)^{\gamma_{12} + \gamma_{23} + \gamma_{13}} (y - y^{-1})^{-2} \\
\left( y^{\gamma_{12} + \gamma_{13} + \gamma_{23}} - y^{\gamma_{12} - \gamma_{23} - \gamma_{13}} - y^{\gamma_{13} + \gamma_{23} - \gamma_{12}} + y^{\gamma_{12} - \gamma_{13} - \gamma_{23}} \right)
\]

agreement with wall-crossing formula's proven by A. Sen (2011)

**NB:** Hard to find fixed points numerically.
Algorithm: Deforming $\gamma_{ij}$

**Problem:** (numerical) determination of fixed points is tedious and time-consuming

**Resolution:** Recursive determination of $g(\{\gamma_i\}, y; \{c_i\})$ using:

1. $\gamma_{ij}$ can be deformed from $\mathbb{Z}$ to $\mathbb{R}$ in Denef equations
2. take a convenient choice $\gamma_{0, ij}$ for $\gamma_{ij}$
3. determine $s_0(p)$
4. study the (dis)appearance of extrema of $W(\{z_i\})$ during the reverse deformation: $s(p) = s_0(p) + \sum_A s_A(p)$

For example:

$N = 2$, $x = z_2 - z_1$, $c_1 < 0$:

$$W(x) = -\gamma_{12} \ln(x) - c_1 x$$

JM, Pioline, Sen (2013)
For quivers **without** closed loops:

- **explicit expression** for \( s(p) \) is obtained:

\[
s(p) = \prod_{k=1}^{N} \Theta(\tilde{\gamma}_{k,k+1} \tilde{c}_k) (-1)^{\sum_{k=1}^{N-1} \Theta(\tilde{\gamma}_{k,k+1})}
\]

where \( \Theta(x) = \text{step function} \)

- **agreement** with Higgs branch result

\[
g(\{\gamma_i\}, y; \{c_i\}) = (-y)^{-\dim C} p(M(\vec{1}_N, \vec{c}_N), -y)
\]

Reineke (2002); JM, Pioline, Sen (2013)
Algorithm: Minimal modification hypothesis

With loops/generic superpotential:
- scaling solutions are possible
- explicit algorithm, recursive in the number of centers
- sum over regular fixed points $\neq$ SU(2) character

Problem: What is the contribution of the scaling fixed point?

1. $y$-dependent: Minimal modification hypothesis:

$$g(\{\gamma_i\}, y; \{c_i\}) = (-1)^{\sum_{i<j} \gamma_{ij} + N-1} \left( \sum_p' s(p) y^{2J_3(p) + p_{\text{scal}}(y)} \right)$$

Determine $p_{\text{scal}}(y)$ iteratively by:
- $g(\{\gamma_i\}, y; \{c_i\})$ is an SU(2) character
- classically $J_3(p_{\text{scal}}) = 0 \Rightarrow \lim_{y \to \infty} \frac{p_{\text{scal}}(y)}{(y - y^{-1})^{N-1}} = 0$

2. $y$-independent: scaling point not distinguishable from single center black hole $\Rightarrow$ include $\Omega_S(\sum_i \gamma_i) \neq 0$
General applicability

Coulomb branch formula computes the invariant $\Omega(\gamma, y; t)$ with as input the $\Omega_S(\gamma)$

Different physical systems lead to different $\Omega_S(\gamma, y)$:
- Quiver quantum mechanics: $\Omega_S(\ell\gamma_i) = \delta_{1,\ell}, i \in V$
  Abelian quivers: Lefshetz hyperplane theorem $\Rightarrow$
  $\Omega_S(\gamma, y) \in \mathbb{N}$
  Bena, Berkooz, De Boer, El-Showk, Van den Bleeken (2012); Lee, Wang, Yi (2012); JM, Pioline, Sen (2013), ...
- Quantum field theories (with line operators) Cordova, Neitzke (2013)
- Supergravity
**Line operators**

**Line operators** $L$ in 4d $\mathcal{N} = 2$ gauge theory:

- Extended along the time direction
- Vacuum expectation value: $\langle L \rangle = \sum_\gamma \overline{\Omega}(L, \gamma) \chi_\gamma$

- $\overline{\Omega}(L, \gamma)$ are determined using traffic rules and satisfy interesting recursion relations

Gaiotto, Moore, Neitzke (2009/10)

- $\overline{\Omega}(L, \gamma)$ are conjectured to be computed by the Coulomb branch formula with $\Omega_S(\gamma) = 0$ for $\gamma \notin \{\text{nodes}\}$

Cordova, Neitzke (2013)

2d line operator on triangulation for $SU(2)$ gauge theory
Quiver mutations

Quiver mutation ⇔ Seiberg duality

Change of gauge groups $U(N_i)$ and number of hypermultiplets $\gamma_{ij}$


Quiver quantum mechanics: $\Omega_S(\ell \gamma_i) = \delta_{1, \ell}, \ i \in V$

Example: mutation on node 2 of 3-node quiver with $c_2 < 0$, $\sum_i c_i = 0$.

\[
\chi(\mathcal{M}(\vec{N}; \vec{c})) = \chi(\mathcal{M}'(\vec{N'}; \vec{c}'))
\]
Generalized quiver mutations

Coulomb branch formula has more general input

\[ \Omega_S(\gamma, y) = \sum_n \Omega_n(\gamma) y^n \in \mathbb{Z}[y, y^{-1}] \]

Generalized mutation symmetry motivated by:

- quiver mutation
- Fermi flip (particle-hole duality)

Andriyash, Jafferis, Denef, Moore (2010)

Conditions:

1. Fermionic particle: \( \Omega_n(\gamma_2) \geq 0 \)
2. \( M_2 = \sum_{\ell \geq 1} \sum_n \ell^2 \Omega_n(\ell \gamma_2) > 0 \)
3. \( \Omega_S(\alpha, y; c) = \left\{ \begin{array}{ll} 
\Omega'_S(\alpha + M_2 \langle \alpha, \gamma_2 \rangle \gamma_2, y; c) & \text{for } \alpha \parallel \gamma_2 \\
\Omega'_S(-\alpha) & \text{for } \alpha \parallel \parallel \gamma_2 
\end{array} \right. \)
Charges transform as:

\[(\gamma_1, \gamma_2, \gamma_3) \rightarrow (\gamma_1 + M_2 \gamma_1 \gamma_2, -\gamma_2, \gamma_3)\]

This induces transformations:

\[(N_1, N_2, N_3) \rightarrow (N_1, M_2 \gamma_1 N_1 - N_2, N_3)\]
\[(c_1, c_2, c_3) \rightarrow (c_1, M_2 \gamma_1 c_1 - c_2, c_3)\]

Proposal:

\[\Omega(\gamma, y; c) = \Omega'(\gamma', y; c')\]

Verified in many cases, but the generalized mutation symmetry remains to be proven.

Puts strong constraints on the \(\Omega_S(\gamma, y)\) JM, Pioline, Sen (2013)
The mathematics of quivers is of fundamental importance for the understanding of BPS states.

BPS bound states in turn provide non-trivial results for quivers:
- Abelianization formula
- Coulomb branch formula
- Single centered invariants
- ... 

Program **CoulombHiggs.m**: 
- **Mathematica** package for Coulomb and Higgs computations 
- available at: www.lpthe.jussieu.fr/ pioline/computing.html