Sheaves on Surfaces and Generalized Appell Functions

Jan Manschot
Trinity College Dublin

GEOQUANT
September 18th, 2015
• Motivation: Yang-Mills path integral and S-duality
• Sheaves on rational surfaces
• Explicit expressions for generating functions
• Generalized Appell functions

Based on:
1407.7785
1109.4861
Classical Yang-Mills theory with gauge group $G$ on 4-manifold $S$:

- Gauge potential: $A \in \Omega^1(S, g)$
- Field strength: $F = dA + A \wedge A \in \Omega^2(S, g)$
- Action: $S[A] = -\frac{1}{g^2} \int_S \text{Tr} F \wedge \star F + \frac{i\theta}{8\pi^2} \int_S \text{Tr} F \wedge F$

For quantitative results beyond the classical theory, consider topologically twisted $\mathcal{N} = 4$ $U(r)$ Yang-Mills on $S$:

Vafa, Witten (1994)

$$h_r(\tau) = \int \mathcal{D}A \mathcal{D}\psi \mathcal{D}X \ e^{-S[A,\psi,X]}$$

with $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$
Path integral localizes on Bogomolny’i-Prasad-Sommerfield solutions:

- minimize $S[A]$, e.g. ASD-instantons: $F = -\ast F$
- instanton number: $\kappa(F) = -\frac{1}{8\pi^2} \int_S \text{Tr} F \wedge F \in \mathbb{Z}$
- $e^{-S[A]} = q^\kappa$ with $q = e^{2\pi i \tau}$

Generating function:

$$\Rightarrow h_r(\tau) = \sum_\kappa \Omega(\kappa) q^\kappa + R(\tau, \bar{\tau})$$

- $\Omega(\kappa)$: Euler number of inst. moduli space $\sim$ BPS-invariant
- $R(\tau, \bar{\tau})$: arises due to reducible connections:

  e.g. $r = 2$ : $A = \begin{pmatrix} A_1 & 0 \\ 0 & -A_1 \end{pmatrix}$
S-duality

Electric-magnetic duality:
Montonen, Olive (1977)

\[
S : \begin{cases} 
(F, *F) \rightarrow (*F, -F) \\
\mathfrak{g} \rightarrow \mathfrak{g}^L \\
\tau \rightarrow -1/\tau
\end{cases}
\]

Translations:

\[
\frac{1}{8\pi^2} \int \text{Tr } F \wedge F \in \mathbb{Z}
\]

\[T : \tau \rightarrow \tau + 1 \text{ is a symmetry of the theory}\]

\[S + T \text{ generate } SL_2(\mathbb{Z}) \text{ S-duality group}\]
Modularity

Instanton equations are invariant under $S$-duality

$\Rightarrow$ modular transformation properties are expected for $h_r(\tau)$:

$$h_r\left(\frac{a\tau + b}{c\tau + d}\right) \sim (c\tau + d)^{-\chi(S)/2} h_r(\tau), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$$

Verification of EM-duality for $U(r = 1, 2)$ YM and complex surfaces using algebraic-geometric techniques

Vafa, Witten (1994)
Computation of $h_r(\tau)$ for rational surfaces is possible using methods for complex surfaces Gottsche, Nakajima, Yoshioka, …

**Donaldson-Uhlenbeck-Yau theorem:**

ASD connections $\iff$ \begin{align*}
&\begin{cases}
- \text{stable holomorphic vector bundles} \\
- \text{reducible connections}
\end{cases}
\end{align*}

Characterized by their **Chern classes** $c_i \in H^{2i}(S, \mathbb{Z})$:

\[ c_1 = \frac{i}{2\pi} \text{Tr } F, \quad c_2 - \frac{1}{2} c_1^2 = \frac{1}{8\pi^2} \text{Tr } F \wedge F \]
Gieseker stability

Let \( C(S) \in H^2(S, \mathbb{R}) \) be the ample cone of \( S \).

Choose polarization (Kähler modulus) \( J \in C(S) \), and define the slope:

\[
\mu_J(E) := \frac{c_1(E) \cdot J}{r(E)}
\]

**Definition:**
A sheaf \( E \) is stable if for every subsheaf \( E' \subsetneq E \), \( \mu_J(E') < \mu_J(E) \) (semi-stable \( \Rightarrow \mu_J(E') \leq \mu_J(E) \)).

Agrees with stability based on central charge in large volume limit:
\[
\lim_{J \to \infty} Z(E, B + iJ)
\]

On varying \( J \), a sheaf \( E \) can cease to be stable or reversely become stable.

This happens across walls of marginal stability (WMS)

**Definition:**
Let \( 0 \subset E' \subset E \), and \( G = E/E' \) and. A wall \( W(E', E) \subset C(S) \) is a codimension 1 subspace of \( C(S) \) such that for \( J_w \in W(E, E') \)

\[
\mu_{J_w}(E') - \mu_{J_w}(E) = 0
\]

but for \( J \notin W(E', E) \)

\[
\mu_{J}(E') - \mu_{J}(E) \neq 0
\]
Invariants

Algebraic-geometric approach can determine more refined invariants than the Euler number.

Let $\mathcal{M}_J(\gamma)$ be the moduli space (stack) of semi-stable sheaves with $\gamma = (r, c_1, c_2)$.

Let $\gamma$ be relatively prime, $J \cdot K_S < 0$ and not on a wall $\Rightarrow \mathcal{M}_J(\gamma)$ is smooth, compact and of expected dimension.

Let $I(\gamma, w; J)$ be the virtual Poincaré function of $\mathcal{M}_J(\gamma)$ (Joyce 2004).

If $\gamma$ is relatively prime, then

$$I(\gamma, w; J) := \frac{w - \dim_{\mathbb{C}} \mathcal{M}_J(\gamma)}{w - w^{-1}} p(\mathcal{M}_J(\gamma), w),$$

where $p(X, s) = \sum_{i=0}^{2\dim_{\mathbb{C}}(X)} b_i s^i$ with $b_i = \dim H^i(X, \mathbb{Z})$. 
Generating function:

\[ H_{r,c_1}(z, \tau; S, J) := \sum_{c_2} \mathcal{I}(\gamma, w; J) q^r \Delta - \frac{r \chi(S)}{24} \]

- discriminant: \( \Delta = \frac{1}{r} \left( c_2 - \frac{r-1}{2r} c_1^2 \right) \)
- modular parameter: \( q = e^{2\pi i \tau}, \quad \tau \rightarrow \frac{a\tau + b}{c\tau + d} \)
- elliptic parameter: \( w = e^{2\pi i z}, \quad z \rightarrow \frac{z}{c\tau + d} \)
Rational ruled surfaces

\[ \Sigma_{\ell} \]

\( f \cong \mathbb{P}^1 \)

\[ \pi : \Sigma_{\ell} \rightarrow \mathbb{P}^1 \]

Rational ruled surface

- \( H_2(\Sigma_{\ell}, \mathbb{Z}) \): generators \( C, f \)
- Intersection numbers: \( C^2 = -\ell, \ f^2 = 0, \ C \cdot f = 1 \)
- Restriction to the rational ruled surface \( \Sigma_1 \) in this talk
  See the refs for other ruled surfaces & their blow-ups/downs
- Ample cone: \( C(\Sigma_1) = \{ m(C + f) + nf; \ m, n > 0 \} \)
Outline of computations

1. Determine invariants for **boundary** polarization

2. Wall-crossing
Conjecture: JM (2011)

\[ H_{r,c_1}(z, \tau; \Sigma_\ell, J_{0,1}) = \begin{cases} 
0 & \text{if } c_1 \cdot f \neq 0 \mod r \\
H_r(z, \tau) & \text{if } c_1 \cdot f = 0 \mod r
\end{cases} \]

where \( H_r(z, \tau) \) is the infinite product:

\[ H_r(z, \tau) := \frac{i (-1)^{r-1} \eta(\tau)^{2r-3}}{\theta_1(2z, \tau)^2 \theta_1(4z, \tau)^2 \ldots \theta_1((2r-2)z, \tau)^2 \theta_1(2rz, \tau)}, \]

with \( \eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \), \( \theta_1(z, \tau) = \sum_{r \in \mathbb{Z} + \frac{1}{2}} (-1)^r q^{\frac{r^2}{2}} w^r \)

Reason: bundles on \( f \cong \mathbb{P}^1 \) are equivalent to direct sums \( \Rightarrow \) unstable for \( c_1 \cdot f \neq 0 \mod r \)

Wall-crossing

Determine \( \mathcal{I}(\gamma, w; J') \) by adding (substracting) HN-filtrations for \( J \) (\( J' \)) which correspond to semi-stable sheaves for \( J' \) (\( J \))

Wall-crossing formula:

\[
\mathcal{I}(\gamma, w; J') = \sum_{\gamma = \gamma_1 + \cdots + \gamma_\ell \atop r_i \geq 1 \ \forall i} S(\{\gamma_i\}, J, J') w - \sum_{i < j} (r_i c_j - r_j c_i) \cdot K_S \prod_{i=1}^{\ell} \mathcal{I}(\gamma_i; w, J)
\]

For all \( i = 1, \ldots, \ell - 1 \), we have either

1. \( \mu_J(\gamma_i) \leq \mu_J(\gamma_{i+1}) \) and \( \mu_{J'}(\sum_{j=1}^{i} \gamma_j) > \mu_{J'}(\sum_{j=i+1}^{\ell} \gamma_j) \),
   or

2. \( \mu_J(\gamma_i) > \mu_J(\gamma_{i+1}) \) and \( \mu_{J'}(\sum_{j=1}^{i} \gamma_j) \leq \mu_{J'}(\sum_{j=i+1}^{\ell} \gamma_j) \),

then \( S = (-1)^k \) with \( k \neq 1 \). is true, else \( S = 0 \).

Joyce (2004)
Explicit expressions for $r = 1, 2 \& \Sigma_1$

**Rank 1:**

$$H_{1,c_1}(z, \tau; J_m, n) = H_1(z, \tau)$$

Gottsche (1990)

**Rank 2:** using wall-crossing formula

$$H_{2, \beta C - \alpha f}(z, \tau; J_m, n) = \delta_{\beta,0} H_2(z, \tau) + H_1(z, \tau)^2 \sum_{(a,b)=-(\alpha,\beta) \mod 2} \frac{1}{2} (\text{sgn}(b - \varepsilon) - \text{sgn}(bn - am - \varepsilon)) w^{-b+2a} q^{\frac{1}{4}b^2 + \frac{1}{2}ab}$$

Indefinite theta function:

Specialize to $J = J_{1,0}$, $\beta = 1$:

\[
H_{2,C-\alpha f}(z, \tau; \Sigma_1, J_{1,0}) = H_1(z, \tau)^2 \sum_{b=1 \mod 2\mathbb{Z}} \frac{w^{2\alpha-b}q^{\frac{1}{4}b^2+\frac{1}{2}\alpha b}}{1 - w^4q^b}
\]


**Specialization** of Appell function $A_\ell(u, v; \tau)$
Appell functions I

Appell function:

\[ A(u, v; \tau) = e^{\pi i u} \sum_{n \in \mathbb{Z}} \frac{(-1)^n e^{2\pi i n v} q^{n(n+1)/2}}{1 - e^{2\pi i u} q^n} \]

- Appell (1884): building blocks of doubly periodic functions
- Ramanujan (1920): mock theta functions
- Zwegers (2002): non-holomorphic completion \( \hat{A}(u, v; \tau) \rightarrow \)
  transforms as weight 1 Jacobi form

Completion: \( \hat{A}(u, v; \tau) = A(u, v; \tau) + \theta_1(z, \tau) R(u - v; \tau) \)

Period integral: \( R(a\tau + b; \tau) = \frac{1}{2i} q^{a^2} e^{-2\pi i a(b + \frac{1}{2})} \int_{-\bar{\tau}}^{i\infty} \frac{g_{a, b}(u)}{\sqrt{-i(z + \tau)}} du \)
Higher level/multi-variable generalizations appearing in mathematical physics:

- **Vacuum character of $\mathcal{N} = 2, 4$ superconformal algebra**
  Eguchi, Taormina (1988)

- **Characters of admissible $\hat{sl}(m|n)$-representations** Kac, Wakimoto (2000), Semikhatov, Taormina, Tipunin (2003)

- **Mirror symmetry of elliptic curves** Polishchuk (1998)

Completion of the generalizations all involve **1-dimensional period integrals**
$H_{r,c_1}(z, \tau; J_{1,0})$ for $r \geq 2$

$H_{r,c_1}(z, \tau; J_{1,0})$ can be determined explicitly for all $(r, c_1)$ using wall-crossing formula of D. Joyce (2004)

JM (2014), Toda (2014)

Example:

$H_{3,0}(z, \tau; J_{1,0}) = H_3(z, \tau) + 2H_1(z, \tau)H_2(z, \tau) \sum_{k \in \mathbb{Z}} \frac{w^{-6k}q^{3k^2}}{1 - w^6q^{3k}}$

$+ H_1(z, \tau)^3 \sum_{k_1, k_2 \in \mathbb{Z}} \frac{w^{-2(k_1+2k_2)}q^{k_1^2+k_2^2+k_1k_2}}{(1 - w^4q^{2k_1+k_2})(1 - w^4q^{k_2-k_1})}$

- Quadratic form for $A_2$ root lattice in numerator
- Two terms in denominator
- Function cannot be written as product of two Appell functions
Generalized Appell functions

Suggests further generalization:

$\implies$ Appell functions with signature $(n_+, n_-)$

Let:

- $\Lambda$: $n_+$-dimensional positive definite lattice
- $\{m_j\}$: set of $n_-$ vectors $\in \Lambda^*$

$$A_{Q,\{m_j\}}(u, v; \tau) = \sum_{k \in \Lambda} \frac{q^{1/2} Q(k) e^{2\pi i v \cdot k}}{\prod_{j=1}^{n_-} (1 - e^{2\pi i u_j q^{m_j \cdot k}})}$$

Modular completion will require higher dimensional period integral
Blow-up formula

\[ \phi : \tilde{S} \to S \]

Bott & Tu

Easy relation between generating functions:

\[ H_{r, \tilde{c}_1}(z, \tau; \phi^* J, \tilde{S}) = B_{r,k}(z, \tau) H_{r,c_1}(z, \tau; J, S), \]

with \( \tilde{c}_1 = \phi^* c_1 - kC_e \) and \( B_{r,k}(z, \tau) \) is a \( A_{r-1} \) theta function.
Identities for Appell functions

Blow-up formula ⇒ relations between $H_{r,1}(z,\tau; J^*, S)$
⇒ relations between Appell functions

Let $S = \mathbb{P}^2$ and $S = \Sigma_1$

Then
\[
\frac{H_{r,C+f}(z,\tau; J_{1,0}, \Sigma_1)}{B_{r,0}(z,\tau)} = \frac{H_{r,f}(z,\tau; J_{1,0}, \Sigma_1)}{B_{r,1}(z,\tau)}
\]

E.g. $r = 2$:
\[
\frac{1}{\theta_3(2z, 2\tau)} \sum_{b \in \mathbb{Z}} \frac{q^{b^2-\frac{1}{4}} w^{2b-3}}{1 - w^{-4} q^{2b-1}} - \frac{1}{\theta_2(2z, 2\tau)} \sum_{b \in \mathbb{Z}} \frac{q^{b^2+b} w^{2b-2}}{1 - w^{-4} q^{2b}}
\]
\[
= -\frac{i\eta(\tau)^3}{\theta_1(4z, \tau) \theta_2(2z, 2\tau)}
\]

For $A_2$ Appell functions

Identities take the following form:

\[
\frac{1}{b_{3,0}(z)} \sum_{k_1, k_2 \in \mathbb{Z}} \frac{w^{-k_1-2k_2} q^{k_1^2+k_2^2+k_1 k_2}}{(1 - w^2 q^{2k_1+k_2})(1 - w^2 q^{k_2-k_1})} \bigg[ 1 - \frac{1}{b_{3,1}(z)} \sum_{k_1, k_2 \in \mathbb{Z}} \frac{w^{-k_1-2k_2-1} q^{k_1^2+k_2^2+k_1 k_2+k_1+k_2+1/3}}{(1 - w^2 q^{2k_1+k_2+1})(1 - w^2 q^{k_2-k_1})} \bigg] \\
+ \frac{2i \eta^3}{\theta_1(2z) b_{3,0}(z)} \sum_{k \in \mathbb{Z}} \frac{w^{-3k} q^{3k^2}}{1 - w^3 q^{3k}} - \frac{i \eta^3}{\theta_1(2z) b_{3,1}(z)} \left( \sum_{k \in \mathbb{Z}} \frac{w^{-3k+1} q^{3k^2-2k+1/3}}{1 - w^3 q^{3k-1}} + \sum_{k \in \mathbb{Z}} \frac{w^{-3k-1} q^{3k^2+2k+1/3}}{1 - w^3 q^{3k+1}} \right) \\
= \frac{\eta^6 \theta_1(z)}{\theta_1(2z)^2 \theta_1(3z) b_{3,0}(z)}.
\]

Proven by Bringman, JM, Rolen (2015)
Conclusions

**Generating functions** of invariants of moduli spaces can be determined explicitly in terms of generalized Appell functions with signature \((n_+, n_-)\)

**Applications:**
- Semi-stable sheaves
- \(q\)-series
- Electric-magnetic duality
- Quantum black hole state counting
- Relation to topological strings on elliptic Calabi-Yau manifolds
- \(
\ldots
\)**