1M01 Calculus Ex sheet 8 solutions.

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(a) $e^{2\ln(x)} = \left(e^{\ln(x)}\right)^2 = x^2$

So $\frac{d}{dx}\left(e^{2\ln(x)}\right) = \frac{d}{dx}\left(x^2\right) = \underline{\underline{2x}}$

(b) $\sqrt{e^{1+2x^3}} = \left(e^{(1+2x^3)}\right)^{\frac{1}{2}} = e^{\frac{1}{2}(1+2x^3)}$

$= e^{\frac{1}{2}+x^3}.$

So $\frac{d}{dx}\left(\sqrt{e^{1+2x^3}}\right) = \frac{d}{dx}\left(e^{\frac{1}{2}+x^3}\right)$

$\boxed{u = \frac{1}{2}+x^3 \quad \frac{du}{dx} = 3x^2}$

$= \frac{d}{du}\left(e^u\right) \cdot \frac{du}{dx}$

$= e^u \cdot 3x^2 = \underline{3x^2 e^{\frac{1}{2}+x^3}}$

(c) $\ln\left(\frac{\sqrt{x}}{\cos(x)}\right) = \ln(\sqrt{x}) - \ln(\cos(x))$

$= \ln\left(x^{1/2}\right) - \ln(\cos(x))$

$= \frac{1}{2}\ln(x) - \ln(\cos(x)).$

So $\frac{d}{dx}\left(\ln\left(\frac{\sqrt{x}}{\cos(x)}\right)\right) = \frac{d}{dx}\left(\frac{1}{2}\ln(x) - \ln(\cos(x))\right)$

$= \frac{1}{2}\frac{d}{dx}\left(\ln(x)\right) - \frac{d}{dx}\ln(u)$ $\quad u = \cos(x)$, $\frac{du}{dx} = -\sin(x)$

$= \frac{1}{2}\cdot\frac{1}{x} - \frac{1}{u}\cdot\frac{du}{dx}$

$= \frac{1}{2x} + \frac{\sin(x)}{\cos(x)} = \underline{\frac{1}{2x} + \tan(x)}.$

② (a) $\int e^{-x/2}+1\,dx = \int e^{-x/2}\,dx + \int 1\,dx$

$u=-x/2$
$du=-\frac{1}{2}dx$
$dx=-2du$

$= -2\int e^{u}\,du + x$

$= -2e^{u} + x + C$

$= -2e^{-x/2} + x + C$

(b) $\int_0^{\ln(2)} \frac{2}{e^{x}}\,dx = \int_0^{\ln(2)} 2e^{-x}\,dx$

$u=-x$
$\frac{du}{dx}=-1$, so
$du=-dx$
$dx=-du$

x	u
0	0
$\ln(2)$	$-\ln(2)$

$= -\int_0^{-\ln(2)} 2e^{u}\,du$

$= \left[-2e^{u}\right]_0^{-\ln(2)}$

$= -2e^{-\ln(2)} - (-2e^{0})$

$= -2\cdot\frac{1}{2} + 2\cdot 1 = -1+2 = 1$

(c) $\int \underbrace{t}_{u}\underbrace{e^{3t}dt}_{dv}$

$u=t$	$dv=e^{3t}$
$du=dt$	$v=\int dv$
	$=\int e^{3t}dt$
	$v=\frac{1}{3}e^{3t}$

$=\int u\,dv = uv - \int v\,du$

$= t\cdot\frac{1}{3}e^{3t} - \int \frac{1}{3}e^{3t}dt$

$= \frac{1}{3}te^{3t} - \frac{1}{3}\int e^{3t}dt$

$= \frac{1}{3}te^{3t} - \frac{1}{3}\cdot\frac{1}{3}e^{3t} + C$

$= \frac{1}{3}te^{3t} - \frac{1}{9}e^{3t} + C$

$\int e^{3t}dt$

$w=3t$
$dw=3dt$
$dt=\frac{1}{3}dw$

$=\frac{1}{3}\int e^{w}dw$

$=\frac{1}{3}e^{w}=\frac{1}{3}e^{3t}$

(3) (a) $P(t) = 10^8 e^{0.02t}$ so $\underline{P(0) = 10^8}$ bacteria.

Generation time: $T = \dfrac{\ln(2)}{k} = \dfrac{\ln(2)}{0.02} = \underline{34.66 \text{ hours}}$.

(b) $P(t) = 10^9 \iff 10^8 e^{0.02t} = 10^9$

$\iff e^{0.02t} = \dfrac{10^9}{10^8} = 10$

$\iff 0.02t = \ln(10)$

$\iff t = \dfrac{\ln(10)}{0.02} = \underline{115.13 \text{ hours}}$.

(4) Half-life is $T = \dfrac{\ln(2)}{k} = 5750$, so $\underline{k = \dfrac{\ln(2)}{5750}}$.

$\dfrac{dN}{dt} = -kN$, so $N(t) = N_0 e^{-kt}$.

Need to find t when $N(t)$ is only $(100-95)\% = 5\%$ of its initial value, N_0:

$\cancel{N_0} e^{-kt} = 0.05 \cancel{N_0}$

$\iff e^{-kt} = 0.05$

$\iff -kt = \ln(0.05)$

$\iff t = -\dfrac{\ln(0.05)}{k}$

$= -\dfrac{\ln(0.05)}{\ln(2)} \times 5750 = \underline{24851 \text{ years}}$