IM01 Summer exam 2010

Section B solutions.

5 (a) $y = 3x - 4$

$\uparrow$ slope: $m = 3$

$y - y_1 = m(x - x_1)$ $\quad m = 3$, $x_1 = 1$, $y_1 = 2$

So $\underline{y - 2 = 3(x - 1)}$.

(b) $\sqrt[4]{16x^3 e^{\ln(x)}} = (16x^3 \cdot x)^{1/4}$ (since $e^{\ln(x)} = x$)

$= (16x^4)^{1/4}$

$= 16^{1/4} \cdot (x^4)^{1/4}$

$= \underline{\underline{2|x|}}$ (2x also acceptable).

(c) $f(x) = x^2 - 3x$

Av rate of change: $\dfrac{y_2 - y_1}{x_2 - x_1}$

$x_1 = -1$, so $y_1 = f(x_1) = (-1)^2 - 3(-1) = 1 + 3 = 4$

$x_2 = 2$, so $y_2 = f(x_2) = 2^2 - 3(2) = 4 - 6 = -2$

So av rate of change is

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{-2 - 4}{2 - (-1)} = \frac{-6}{3} = \underline{-2}.$$

6 (a)(i) $\frac{d}{dx}\left(\frac{\sin(x)}{1+x}\right) \underset{\uparrow}{=} \frac{(1+x)\cos(x) - \sin(x)\cdot 1}{(1+x)^2} = \underline{\frac{(1+x)\cos(x) - \sin(x)}{(1+x)^2}}$

by quotient rule

(ii) $\frac{d}{dx}\left(5 - 6e^{1+x^2} + x\ln(x)\right) = 0 - 6\frac{d}{dx}\left(e^{1+x^2}\right) + \frac{d}{dx}\left(x\ln(x)\right)$

and $\frac{d}{dx}\left(e^{1+x^2}\right) = \frac{d}{du}\left(e^u\right)\frac{du}{dx}$, $\quad u = 1+x^2$; $\frac{du}{dx} = 2x$

(chain rule)

$= e^u \cdot 2x$

$= 2x e^{1+x^2}$;

and $\frac{d}{dx}\left(x\ln(x)\right) \underset{\uparrow}{=} x \cdot \frac{1}{x} + 1 \cdot \ln(x) = 1 + \ln(x)$

product rule

So answer is $-6 \cdot 2xe^{1+x^2} + 1 + \ln(x) = \underline{-12xe^{1+x^2} + 1 + \ln(x)}$

(iii) $\frac{d}{dx}\left(\left(x^2 + \sin(2\pi x)\right)^{0.3}\right)$

take $u = x^2 + \sin(2\pi x)$

(using chain rule)

then $\frac{du}{dx} = 2x + 2\pi\cos(2\pi x)$

$= \frac{d}{du}\left(u^{0.3}\right) \cdot \frac{du}{dx}$

$= 0.3u^{-0.7} \cdot \left(2x + 2\pi\cos(2\pi x)\right)$

$= \underline{0.6\left(x + \pi\cos(2\pi x)\right) \cdot \left(x^2 + \sin(2\pi x)\right)^{-0.7}}$

⑥(b) $\int e^x - 4x\sin(x^2)\,dx$

$= \int e^x\,dx - \underbrace{\int 4x\sin(x^2)\,dx}$ → take $u = x^2$
$du = 2x\,dx$
$\therefore 2du = 4x\,dx$

$= e^x - \int 2\sin(u)\,du$

$= e^x + 2\cos(u) + C$

$= e^x + 2\cos(x^2) + C$

⑦ $f(x) = 12x - x^3 + 10.$

(a) $f'(x) = 12 - 3x^2$

So $f'(x) = 0 \iff 12 - 3x^2 = 0$

$\iff 3x^2 = 12$

$\iff x^2 = 4$

$\iff \underbrace{x = 2}_{\text{in } [0,3]}$ or $\underbrace{x = -2}_{\text{not in } [0,3]}$

Since $f'(x)$ exists for all $x \in [0,3]$, only have to consider $x = 2$, and the endpoints $x = 0$ & $x = 3$.

x	$f(x) = 12x - x^3 + 10$	
0	$12\cdot 0 - 0^3 + 10 = 10$	← abs. min at $(0,10)$
2	$12\cdot 2 - 2^3 + 10 = 24 - 8 + 10 = 26$	← abs. max at $(2,26)$
3	$12\cdot 3 - 3^3 + 10 = 36 - 27 + 10 = 19$	

(b) $y_{av} = \frac{1}{x_2 - x_1}\int_{x_1}^{x_2} f(x)\,dx = \frac{1}{3-0}\int_0^3 12x - x^3 + 10\,dx$

$= \frac{1}{3}\left[6x^2 - \frac{x^4}{4} + 10x\right]_0^3 = \frac{1}{3}\left(\left(6\cdot 3^2 - \frac{3^4}{4} + 10\cdot 3\right) - (0)\right)$

$= 6\cdot 3 - \frac{3^3}{4} + 10 = \underline{\underline{21.25}}$.

(8) $P(t) = (3\times10^5)\, e^{0.04t}$

(a) "introduced" at $t=0$:

$P(0) = 3\times10^5 e^0 = \underline{3\times10^5}$ bacteria

(b) Gen. time is $T = \dfrac{\ln 2}{k}$ for $P(t) = P_0 e^{kt}$

Here, $k = 0.04$.

So $T = \dfrac{\ln 2}{0.04} \approx \underline{17.3 \text{ hours}}$

(c) Want $\boxed{P'(24)}$.

$P'(t) = \frac{d}{dt}\left((3\times10^5)e^{0.04t}\right)$

$= 3\times10^5 \times 0.04 \times e^{0.04t} = 12000 e^{0.04t}$

So $P'(24) = 12000 e^{0.04\times 24} \simeq \underline{31340.4 \text{ bacteria/hr}}$

(d) $P(t) = 2\times10^6$: need to solve for t.

$\Leftrightarrow (3\times10^5)e^{0.04t} = 2\times10^6$

$\Leftrightarrow e^{0.04t} = \dfrac{2\times10^6}{3\times10^5} = \dfrac{20}{3}$ (...now take ln of both sides...)

$\Leftrightarrow 0.04t = \ln\left(\frac{20}{3}\right)$

$\Leftrightarrow t = \dfrac{1}{0.04}\ln\left(\frac{20}{3}\right) \simeq \underline{47.4 \text{ hours}}$