1M01 Summer 2009
Section B solutions

(5) (a) $25^\circ = \frac{25^\circ}{180^\circ} \cdot \pi = \underline{\underline{\frac{5}{36}\pi}}$

(b) Slope: $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-3-1}{2-1} = \underline{\underline{-4}}$ $\left(\begin{matrix} x_1 = 1, y_1 = 1 \\ x_2 = 2, y_2 = -3 \end{matrix}\right)$

So equation is $y - y_1 = m(x - x_1)$:

$$\underline{y - 1 = -4(x-1)}.$$

(c) $y = \frac{\ln(x)}{1+x}$

Need to find slope @ $x=1$: $\left.\frac{dy}{dx}\right|_{x=1}$.

$\frac{dy}{dx} = \frac{(1+x)\cdot\frac{1}{x} - \ln(x)\cdot 1}{(1+x)^2}$ (by quotient rule)

So $\left.\frac{dy}{dx}\right|_{x=1} = \frac{(1+1)\cdot\frac{1}{1} - \ln(1)\cdot 1}{(1+1)^2} = \frac{2-0}{2^2} = \underline{\underline{\frac{1}{2}}}$

(6) (a) $\int_0^1 x(x^2+1)^{15}\,dx$

Substitution:
take $u = x^2+1$
then $du = 2x\,dx$,
$x\,dx = \frac{1}{2}du$.

x	$u = x^2+1$
0	1
1	2

$= \int_1^2 \frac{1}{2} u^{15}\,du$

$= \left[\frac{1}{2}\cdot\frac{u^{16}}{16}\right]_1^2 = \frac{2^{16}}{32} - \frac{1}{32} = \underline{\underline{\frac{1}{32}(2^{16}-1)}}$

(b) Min: $(x+1)^2+y^2$ with $x+y=1$.

$\iff y=1-x$.

So need to Min: $(x+1)^2+(1-x)^2$ for x a real number.

Observe: if $x>9$ then $\underbrace{(x+1)^2}_{>10^2}+\underbrace{(1-x)^2}_{\geq 0}>10^2=100$

and if $x<-9$ then $\underbrace{(x+1)^2}_{\geq 0}+\underbrace{(1-x)^2}_{>10^2}>10^2=100$ (*)

Seek min of $(x+1)^2+(1-x)^2$ for x in $[-9,9]$.

derivative is $2(x+1)-2(1-x)=4x=0 \iff x=0$,

so need to check $x=0$, and endpoints:

x	$f(x)=(x+1)^2+(1-x)^2$
-9	$(-9+1)^2+(1+9)^2=164$
0	$(0+1)^2+(1-0)^2=2$
9	$(9+1)^2+(1-9)^2=164$

$\leftarrow$ min over $[-9,9]$ is at $x=0$, & min value is $\boxed{2}$.

(This is the min value over <u>all</u> real numbers, by (*)).

[Remark: This is a cumbersome way to do this question, because we didn't cover finding max/min over <u>all</u> real numbers this year (2010-11), just over intervals $[a,b]$.]

(7) (a) $f(x) = 1 + 2\cos(\pi x)$.

av rate of change is $\dfrac{y_2 - y_1}{x_2 - x_1}$

$x_1 = 0$; $y_1 = f(0) = 1 + 2\cos(\pi \cdot 0) = 3$

$x_2 = 3$; $y_2 = f(3) = 1 + 2\cos(\pi \cdot 3) = 1 + 2 \cdot (-1) = -1$.

So av rate of change is $\dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{-1-3}{3-0} = \dfrac{-4}{3}$

(b) Want: $f'(3)$.

$$f'(x) = \frac{d}{dx}\left(1 + 2\cos(\pi x)\right) = 0 + 2 \cdot \pi \cdot (-\sin(\pi x))$$

$$= -2\pi \sin(\pi x).$$

So $f'(3) = -2\pi \underbrace{\sin(\pi \cdot 3)}_{=0} = 0$.

(c) $$y_{av} = \frac{1}{x_2 - x_1}\int_{x_1}^{x_2} f(x)\,dx$$

$$= \frac{1}{3-0}\int_0^3 1 + 2\cos(\pi x)\,dx$$

$$= \frac{1}{3}\left[x + \frac{2}{\pi}\sin(\pi x)\right]_0^3$$

$$= \frac{1}{3}\left(\left(3 + \frac{2}{\pi}\underbrace{\sin(\pi \cdot 3)}_{=0}\right) - \left(0 + \frac{2}{\pi}\underbrace{\sin(\pi \cdot 0)}_{=0}\right)\right)$$

$$= \frac{1}{3}(3) = 1$$

(8) (a) [We didn't discuss continuity this year].

(b) $h(t) = 1.6$? Solve for t.

Need $t > 100$. (since for $0 \le t \le 100$, $h(t) = 1.1 \neq 1.6$).

So solve $1.1 + e^{1-0.01t} = 1.6$:

$$\iff e^{1-0.01t} = 0.5 \quad \text{(... now take ln of both sides ...)}$$

$$\iff 1 - 0.01t = \ln(0.5)$$

$$\iff 0.01t = 1 - \ln(0.5)$$

$$\iff t = 100(1 - \ln(0.5)) \simeq \underline{169.31 \text{ s}}.$$

(c) If $H(t)$ = number of times the man's heart beats up to time t

then $h(t) = H'(t)$ (since $h(t)$ measures heart rate, or rate of change of $H(t)$)

so

number of times heart beats between $t=0$ & $t=200$

$$= H(200) - H(0)$$

$$= \int_0^{200} h(t)\,dt \quad \text{(by FTC 2) (see section 3.3 of notes)}$$

$$= \int_0^{100} h(t)\,dt + \int_{100}^{200} h(t)\,dt$$

$$= \int_0^{100} 1.1\,dt + \int_{100}^{200} 1.1 + e^{1-0.01t}\,dt$$

$$= \underline{110 + \int_{100}^{200} 1.1 + e^{1-0.01t}\,dt}.$$