7 Exponential and logarithmic functions

7.1 The exponential function

Textbook: Section 4.1

A function $f(x)$ has *exponential growth* if $\frac{f(x+1)}{f(x)}$ is a constant $k > 1$, not depending on x. This means that for any x, if you increase x by 1, you multiply the y-value $f(x)$ by k to get the new value, $f(x+1)$.

For example, $y = 2^x$ is has exponential growth, with $k = 2$.

Example 7.1.1. The population $P(t)$ of a culture of bacteria t hours after the start of an experiment is given by

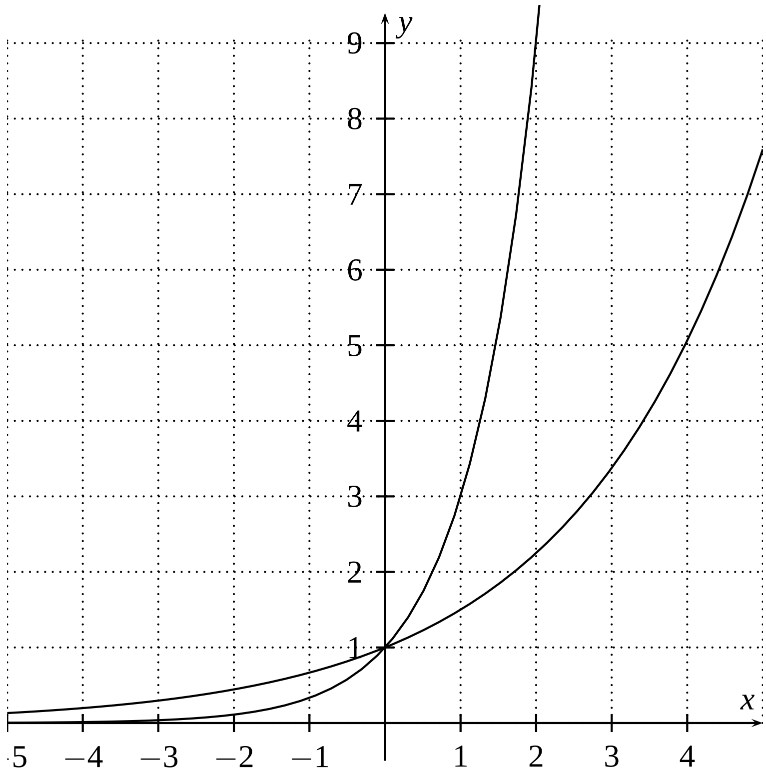
$$P(t) = 1000 \cdot 1.1^t.$$

(a) What is the initial population?

(b) Explain why $P(t)$ has exponential growth by showing that the population increases by 10% every hour.

(c) A different sample of bacteria has an initial population of 500 and increases by 25% every hour. Suggest a function to model this population.

Example 7.1.2. The sketch below shows the graphs of $y = a^x$ for $a = 1.5$ and $a = 3$. Add the graph of $y = 2^x$. What happens to the slope of the tangent line to each of these curves at $x = 0$ as a increases?



Definition 7.1.3. The constant e is the real number with the property that the graph of $y = e^x$ has a tangent line of slope 1 at $x = 0$. The function $y = e^x$, sometimes written $y = \exp(x)$, is called *the exponential function*.

From the discussion above, we see that e is between 2 and 3. In fact,

$$e = 2.7182818\ldots.$$

Your calculator will tell you this if you ask it to compute `exp(1)` or $\mathtt{e}^1$.

The laws of exponents we discussed on page 43 apply in particular to the function $f(x) = e^x$:

Theorem 7.1.4 (Power laws for e^x)**.** *For any real numbers x and y,*

1. $e^x \cdot e^y = e^{x+y}$ *and* $\dfrac{e^x}{e^y} = e^{x-y}$
2. $(e^x)^y = e^{xy}$

Example 7.1.5. Simplify $e^x(e^{-x})^2$.

Example 7.1.6. Simplify $e^2 \cdot \sqrt{\dfrac{(e^{3x})^5}{e^{-x}}}$.

Theorem 7.1.7 (The derivative of e^x)**.** $\dfrac{d}{dx}(e^x) = e^x$.

By the chain rule, we also have:

Theorem 7.1.8 (The derivative of e^u)**.** *If u depends on x, then* $\dfrac{d}{dx}(e^u) = e^u \cdot \dfrac{du}{dx}$.

Example 7.1.9. Find the derivatives of:

(i) e^{-2x} (ii) $(e^{\sin(x)})^2$ (iii) $\cos(e^x)$ (iv) $e^{\sqrt{x^2+1}}$.

Theorem 7.1.10 (The antiderivative of e^x)**.** $\int e^x\,dx = e^x + C.$

Example 7.1.11. Find $\int_0^2 e^x\,dx$.

Example 7.1.12. What is $\int e^{-2x}\,dx$?

Example 7.1.13. What is $\int xe^{-x^2}\,dx$?

Example 7.1.14. What is $\int x^3e^{-x^2}\,dx$?

7.2 Logarithms

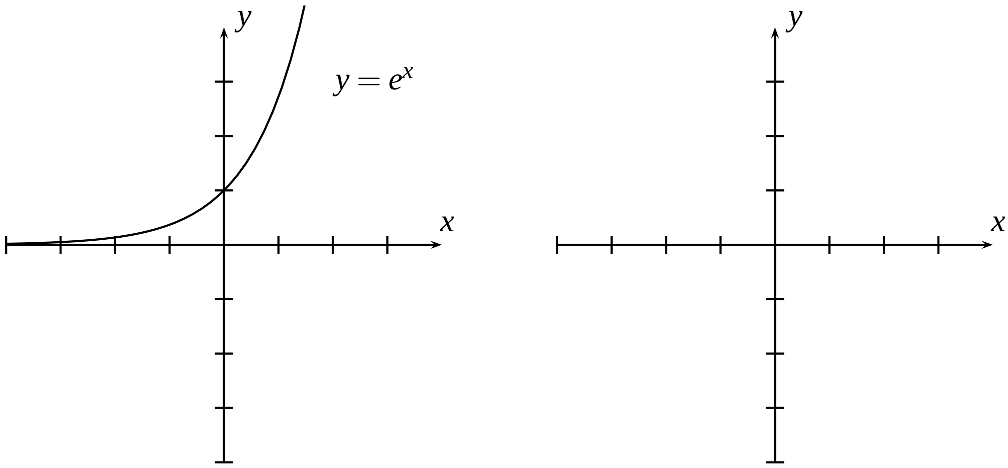
Textbook: Section 4.2

Definition 7.2.1. If $x > 0$ then the *natural logarithm* of x is the real number y with $e^y = x$. We write $y = \ln(x)$ or $y = \ln x$. We have

$$y = \ln(x) \iff e^y = x \qquad \text{for any real number } y \text{ and any } x > 0.$$

This defines a function $y = \ln(x)$ which is defined for $x > 0$, and is undefined for $x \leq 0$.

Example 7.2.2. Given the graph of $y = e^x$ on the left, swap x and y to get the graph of $x = \ln(x)$. What are $\ln(1)$ and $\ln(2)$? What are $\ln(-1)$ and $\ln(0)$?



Example 7.2.3. Simplify $e^{\ln(x)}$ and $e^{2\ln(x)}$.

Example 7.2.4. Simplify $\ln(e^x)$ and $\ln(e^{-3x^2} \cdot e^4)$.

Example 7.2.5. For which real numbers x is the function $f(x) = 2 + \frac{1}{2}\ln(3 - x)$ defined? What are $f(6)$ and $f(-6)$?

Using the properties of e^x discussed in the previous section, we can deduce some properties of $\ln(x)$.

Theorem 7.2.6 (Properties of $\ln(x)$)**.** *For any positive numbers a and b and any real number k,*

1. $\ln(ab) = \ln(a) + \ln(b)$ *and* $\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$
2. $\ln(a^k) = k \cdot \ln(a)$
3. $\ln(e^k) = k$ *(so, for example,* $\ln(e) = \ln(e^1) = 1$ *and* $\ln(1) = \ln(e^0) = 0$*).*
4. $e^{\ln(a)} = a$

Why is $\ln(x)$ an important function important for scientists?

One reason $\ln(x)$ is important is that it lets you solve equations where the unknown variable is in a power, such as $2^x = 10$.

Here is another reason you might be familiar with. In many situations, one quantity might be directly proportional to a power of another. So if x and y are the quantities of interest, then you expect that $y = cx^m$ for some power m and a positive constant c. To find the values of the constants c and m, you could imagine doing an experiment to get some values for x, y. Then take ln of all of your values: if we write $Y = \ln(y)$ and $X = \ln(x)$ then we have

$$y = kx^s \implies \ln(y) = \ln(kx^s) = \ln(c) + m\ln(x) \implies Y = mX + C$$

where $C = \ln(c)$. So if you plot Y against X, you expect to get a straight line with slope m and Y-intercept C. By measuring m and C (and using $c = e^C$) you have found the constants in the formula you're really interested in, $y = cx^m$.

This is why "log-log" plots are so useful.

Example 7.2.7. Solve the equation $2^x = 10$.

Example 7.2.8. Solve the equation $\ln(5x^{-2/3}) = \ln(25)$.

If $y = \ln(x)$ then $e^y = x$, so differentiating both sides of this equation with respect to x using the chain rule gives

$$\frac{d}{dx}(e^y) = \qquad\qquad\qquad \text{so} \quad \frac{dy}{dx} =$$

Theorem 7.2.9 (The derivative of $\ln(x)$)**.**

$$\frac{d}{dx}(\ln(x)) = \frac{1}{x} \quad \textit{for } x > 0.$$

Applying the chain rule, we deduce:

Theorem 7.2.10. *If u depends on x, then*

$$\frac{d}{dx}(\ln(u)) = \frac{1}{u} \cdot \frac{du}{dx}.$$

Example 7.2.11. Find the derivatives of $\ln(4x)$ and $\ln(\sqrt{x^2+1})$.

Example 7.2.12. (i) Let a be a positive constant with $a \neq 1$. Use the equation $a = e^{\ln(a)}$ to write a^x in terms of e^x, and hence find the derivative $\frac{d}{dx}(a^x)$.

(ii) What is the slope of the tangent line to the curve $y = 2^x$ at $x = 0$?

7.3 The uninhibited growth model $\frac{dP}{dt} = kP$

Textbook: Section 4.3

Let $P = P(t)$ be the size of a population at time t. If the population growth rate is directly proportional to the population size, then there is a constant $k > 0$ with

$$\frac{dP}{dt} = kP.$$

[Another way to write this is $P'(t) = kP(t)$.]

Theorem 7.3.1 (The solutions to $\frac{dy}{dx} = ky$)**.** *Let k be any real constant. The functions $y = y(x)$ which satisfy*

$$\frac{dy}{dx} = ky$$

are precisely the functions which can be written

$$y = ce^{kx}$$

where c is a constant. In fact, c is the value of y at $x = 0$.

The equation $\dfrac{dy}{dx} = ky$ is called a *differential equation* since it involves the derivative of a function. The theorem tells us all the functions $y = y(x)$ which satisfy the differential equation. These functions are the *solutions* to this differential equation.

Example 7.3.2. Find a function $f(x)$ so that $f'(x) = 4f(x)$ and $f(0) = 25$.

Example 7.3.3. Show that the function $P(t) = 1000 \times 1.1^t$ from Example 7.1.1 can be rewritten in the form $P(t) = ce^{kt}$ for some constants c and k, and write down a differential equation which P satisfies.

Returning to our population model: $\frac{dP}{dt} = kP$ for some constant $k > 0$. The theorem tells us that

$$P(t) = P_0 e^{kt}$$

for some constant P_0.

We say that $P(t) = P_0 e^{kt}$ *grows exponentially*.

Definition 7.3.4. Suppose that $P(t)$ grows exponentially with $P(t) = P_0 e^{kt}$ for some constants P_0 and $k > 0$. The *generation time*, or *doubling time* T is the time it takes P to double in size from its initial value P_0.

In summary:

- $T = \frac{\ln(2)}{k}$
- whenever t increases by the generation time T, the value of $P(t)$ doubles.

Example 7.3.5. If $P(t) = 10 \cdot e^{5t}$, what is the generation time?

Example 7.3.6. A population of size $P(t)$ grows exponentially from an initial population of 200. If the population doubles every 7 days, find an equation for $P(t)$.

Example 7.3.7. A rabbit population $P(t)$ grows according to

$$P(t) = 400 \times 1.03^{2t}$$

where t is the time, in months, since the population was first measured. Find the doubling time for this population, and find the population growth rate after one month.

7.4 Exponential decay

Textbook: Section 4.4

Radioactive isotopes decay exponentially. If $N(t)$ is the quantity of a given radioactive isotope in a sample, then the rate of decay, $-\frac{dN}{dt}$, is directly proportional to $N(t)$. So there is a constant $k > 0$ with

$$\frac{dN}{dt} = -kN.$$

By Theorem 7.3.1, we have $N(t) = N_0 e^{-kt}$ for some constant N_0.

Definition 7.4.1. Suppose that $N = N(t)$ decays exponentially according to $N(t) = N_0 e^{-kt}$ for some constant $k > 0$. The *half life* T is the time it takes N to halve in size from its initial value N_0.

Just as for the doubling time, it's easy to see that:

- $T = \frac{\ln(2)}{k}$
- whenever t increases by the half life T,
  the value of $N(t)$ halves.

Example 7.4.2. (i) If $N(t) = 8 \times 10^{12} \times e^{-0.003t}$ where t is time in days, what is the half life?

(ii) When is $N(t)$ equal to 2×10^{12}?

(iii) How long will it take before $N(t)$ is less than 10^6?

Example 7.4.3. The radioactive element Carbon-14 has a half-life of 5750 years. The amount $N(t)$ of Carbon-14 in a sample at time t, measured in years, decays according to the differential equation

$$\frac{dN}{dt} = -kN$$

where k is some positive constant.
(i) State an equation for $N(t)$, and sketch a graph showing the behaviour of $N(t)$ as t increases. Mark the half-life of $N(t)$ on your sketch.

(ii) Which of the following three statements is true, and which is false? Explain your answers.

1. The quantity of Carbon-14 in a sample is directly proportional to $1/t$.

2. The rate of decay of Carbon-14 in a sample is directly proportional to $N(t)$.

3. After $2 \times 5750 = 11,500$ years, all of the Carbon-14 in a sample will have decayed.

(iii) Compute the age of a sample which has lost 60% of its Carbon-14.