6 Further differentiation and integration techniques

Here are three more rules for differentiation and two more integration techniques.

6.1 The product rule for differentiation

Textbook: Section 2.7

Theorem 6.1.1 (The product rule)**.** *If u and v are functions of x, then*

$$\frac{d}{dx}(u \cdot v) = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}.$$

Example 6.1.2. Check that this works if $u = x^3$ and $v = x$.

Example 6.1.3. Find the derivative of $(x^2+3)\sin(x)$.

Example 6.1.4. If $f(x) = 3x\sqrt{x^8+2}$, compute the slope of the tangent line to $y = f(x)$ at $x = 0$.

Example 6.1.5. Find $\dfrac{dy}{dx}$ if $y = 3x + \sin(\pi x)\cos(x^2 - x)$.

Warning. The "rule" $\dfrac{d}{dx}(u \cdot v) = \dfrac{du}{dx} \cdot \dfrac{dv}{dx}$ is *wrong*. Use the product rule instead!

6.2 The quotient rule for differentiation

Textbook: Section 2.7

Theorem 6.2.1 (The quotient rule)**.** *If u and v are functions of x, then*

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \cdot \frac{du}{dx} - u \cdot \frac{dv}{dx}}{v^2}.$$

Example 6.2.2. Check that this works if $u = x^3$ and $v = x$.

Example 6.2.3. Find the derivative of $\dfrac{\sin(x)}{x^2+3}$.

Example 6.2.4. If $f(x) = \dfrac{3x}{\sqrt{x^8+2}}$, find the slope of the tangent line to $y = f(x)$ at $x = 0$.

Example 6.2.5. The function tan is defined by $\tan(t) = \dfrac{\sin(t)}{\cos(t)}$. What is $\dfrac{d}{dt}\big(\tan(t)\big)$?

Warning. The "rule" $\dfrac{d}{dx}(u/v) = \dfrac{du}{dx}\Big/\dfrac{dv}{dx}$ is *wrong*. Use the quotient rule instead!

6.3 The chain rule for differentiation

Textbook: Section 2.8

The chain rule is a rule for differentiating an expression formed by applying one function after another. We've already used two special cases of the chain rule: the extended power rule, Theorem 4.3.4, and the chain rule for trigonometric functions, Theorem 5.5.3.

Theorem 6.3.1 (The chain rule). *If $y = f(u)$ where u is a function of x then*

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

To use the chain rule, you have to decide which part of the formula to call u.

For example, suppose that we want to differentiate $f(x) = \sin(x^2)$. This is the same as $f(u)$ where $u = x^2$ and f is the sin function, so we can use the chain rule.

Example 6.3.2. Let $y = \sin(x^2)$. What is $\frac{dy}{dx}$?

Example 6.3.3.

(a) Find $f'(x)$ if $f(x) = \cos(3x - 1)$.

(b) What is $\frac{dy}{dt}$ if $y = 5\cos(2\pi t) + 12$?

(c) Compute $\frac{d}{dx}\left(\sqrt[3]{2 + \frac{1}{x}}\right)$.

Example 6.3.4. Let $f(x) = \tan(2x - \frac{\pi}{4})$. What is $f'(x)$?

Example 6.3.5. Bacterial growth r is related to temperature T by the equation

$$r = k(T - T_0)^2$$

where T_0 and k are constants.

(a) Find $\dfrac{dr}{dT}$.

(b) If T varies according to the formula $T(t) = T_0 + \frac{1}{3}t^3$ where t is time in hours, find $\dfrac{dr}{dt}$ in terms of t, k and T_0.

(c) What is the rate of change of r with respect to t when $t = 1$?

6.4 Integration by substitution

Textbook: Section 5.5

Theorem 6.4.1 (Integration by substitution)**.** *If* $f = f(u)$ *and* $u = u(x)$ *then*

$$\int f(u) \cdot \frac{du}{dx}\, dx = \int f(u)\, du.$$

We can check that this is true using the chain rule. Indeed:

$$\frac{d}{dx} \int f(u)\, du =$$

It appears that in Theorem 6.4.1, we are simply replacing $\frac{du}{dx}\, dx$ with du. This is useful notation, and we will write

$$du = \frac{du}{dx}\, dx.$$

Example 6.4.2. Find $\int 2x \sin(x^2)\, dx$.

As usual, you can (and should) check your answer by differentiating.

Example 6.4.3. What is $\int \frac{7}{x^4} \sqrt{2 + \frac{1}{x^3}}\, dx$?

Example 6.4.4. Compute the following antiderivatives.

(a) $\int (5-x)^{4/3}\,dx$

(b) $\int \frac{r^2}{2}\sqrt{r^3+10}\,dr$

If we can find the indefinite integral of $f(x)$ by substitution then we can compute definite integrals of $f(x)$ such as $\int_a^b f(x)\,dx$. There are two methods, and you can use whichever you prefer. You just have to remember that the limits a and b are x-values, so you should either write everything in terms of x before subbing them in, or change them into u-values and sub them into a formula with u's in it.

The second method is often quicker.

Example 6.4.5. Compute the definite integrals:

(a) $\int_0^1 t\sin(\pi t^2)\,dt$

(b) $\int_0^{\pi/4} \frac{\sin(x)}{\sqrt{\cos(x)}}\,dx$

6.5 Integration by parts

Textbook: Section 5.6

Theorem 6.5.1 (Integration by parts)**.** *If u and v are functions of x, then*

$$\int u\,dv = uv - \int v\,du.$$

Example 6.5.2. Find the indefinite integral $\int x\cos(x)\,dx$, and check your answer by differentiation.

To use this formula, you must:

1. Split the function being integrated as a product of two things, call one of them u and the other dv.
2. Compute $\frac{du}{dx}$ by differentiating u; then $du = \frac{du}{dx}dx$.
3. Compute v from dv by finding the antiderivative of dv.
4. Sub everything into the formula for integration by parts. Now you'll have another integral, $\int v\,du$ to compute, using whatever method you choose (recognising it as a standard antiderivative, or using substitution, or another integration by parts, or...)

To make this process work, in step 1 you have to make an inspired guess of how to write the function being integrated as a product $u \cdot dv$, in such a way that you can:

- differentiate u (usually not hard)
- integrate dv (which can be more difficult)
- *and* you can do the integral $\int v\,du$.

Example 6.5.3. We'll meet the function $\ln(x)$ soon. Its derivative is $\frac{1}{x}$. Use this information to find $\int \ln(x)\,dx$.

To find definite integrals with integration by parts, we "keep the limits around" on the right hand side as follows. Remember that the limits a,b in this formula are always x-values, and u and v depend on x.

$$\int_a^b u\,dv = \Big[uv\Big]_a^b - \int_a^b v\,du.$$

Example 6.5.4. What is $\displaystyle\int_0^1 \frac{x^3}{\sqrt{1+x^2}}\,dx$?