5 Trigonometric functions

Trigonometric functions are functions like sin and cos whose input values represent *angles*. They are *periodic*: they repeat themselves at regular intervals. This makes them very useful for modelling periodic phenomena such as seasonal temperatures, tidal heights etc.

5.1 Angles and rotations

Textbook: Section 1.4

An angle is a measure of rotation. We will need units to measure rotations. We use positive numbers to represent anticlockwise rotations and negative numbers to represent clockwise rotations.

One common choice of unit is the degree, where 180° is the smallest counterclockwise rotation needed to turn back to front.

So $2 \times 180^\circ = 360^\circ$ is a complete rotation,

$\frac{1}{2} \times 180^\circ = 90^\circ$ is a right angle and $\frac{1}{3} \times 180^\circ = 60^\circ$ is the angle at each vertex of an equilateral triangle.

In calculus, it's more convenient to choose different units (we'll see why later). These are called radians. Instead of 180°, we measure the rotation needed to turn back to front as π radians.

We usually don't write any units and just assume that we're using radians. Then $2 \times \pi = 2\pi$ is a complete rotation, $\frac{1}{2}\pi$ is a right angle and the angle in an equilateral triangle is $\frac{1}{3}\pi$.

Warning. The symbol π is *not* a symbol for a unit, such as $^\circ$ or kg. It is a constant number: $\pi = 3.14159265\ldots$. This will be important later on when we find the derivative of a function like $\sin(\pi x)$. The units for angle measure in radians is "radians", for example

$$\pi \text{ radians} = 180^\circ.$$

But we often don't bother to write "radians", so we write

$$\pi = 180^\circ.$$

Of course, this *doesn't* mean that $\pi = 180$, because π is the constant $3.14159265\ldots$ which is very different to the constant 180. Instead, this means that the angle π radians is equal to the angle 180°. This is like saying that the length 1 m is equal to the length 100 cm; this doesn't mean that $1 = 100$.

Theorem 5.1.1 (Converting angles from radians to degrees and vice versa)**.**

$$\frac{\textit{radian measure}}{\pi} = \frac{\textit{degree measure}}{180^\circ}.$$

What makes this theorem work? Both sides of the equation simply count the number of "about-turns" that the angle represents.

Rearranging this equation, we can convert between radians and degrees:

$$\text{radian measure} = \frac{\text{degree measure}}{180^\circ} \times \pi \quad \text{and}$$

$$\text{degree measure} = \frac{\text{radian measure}}{\pi} \times 180^\circ.$$

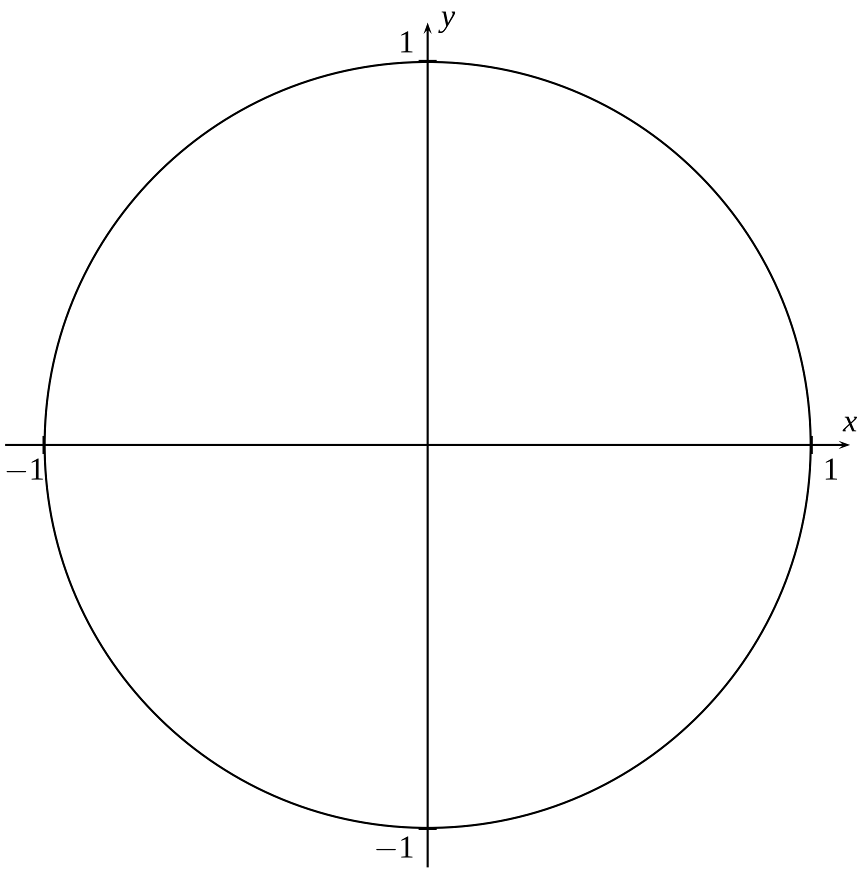
Example 5.1.2.

(a) What is -30° in radians?
(b) What is $5\pi/4$ in degrees?
(c) Convert 400° into radians.
(d) Convert 1 radian into degrees.

5.2 Trigonometric functions and the unit circle

Textbook: Section 1.5

The *unit circle* is the circle of radius 1 in the x-y plane with centre $(0,0)$.



Definition 5.2.1. If t is an angle measured in radians, let P_t be the point on the unit circle which you get by rotating $(1,0)$ by the angle t around $(0,0)$. Then we define

$$\cos(t) = \text{the } x\text{-coordinate of } P_t \quad \text{and} \quad \sin(t) = \text{the } y\text{-coordinate of } P_t.$$

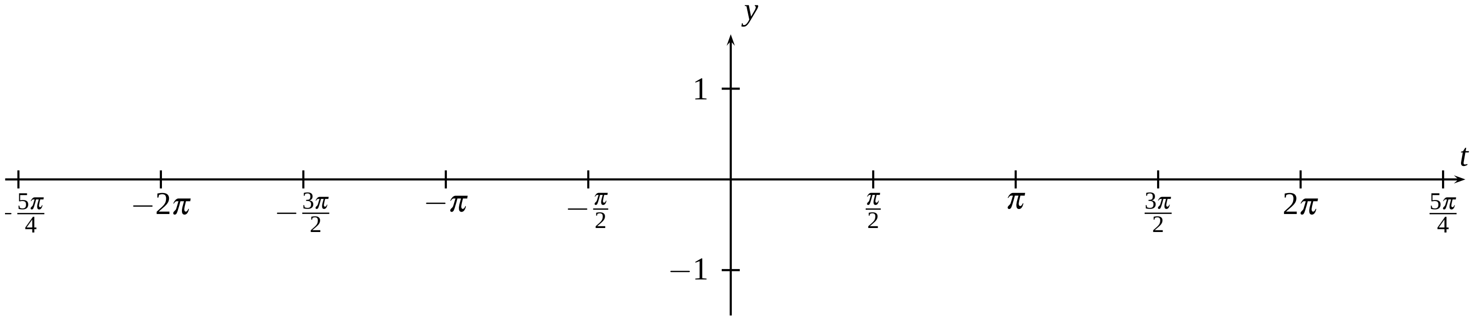
Example 5.2.2. Explain why $\cos(0) = 1$ and $\sin(0) = 0$.

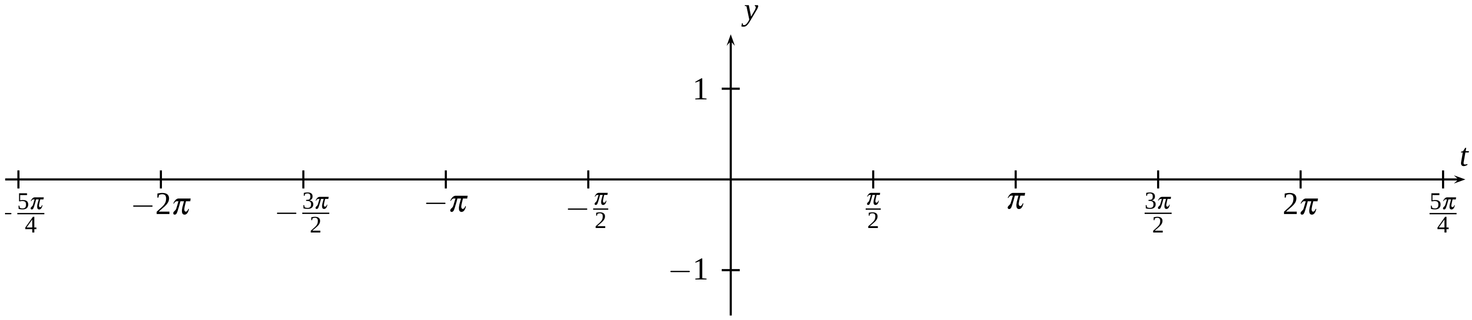
Example 5.2.3. Use the unit circle to calculate $\cos(-\frac{\pi}{2})$ and $\sin(-\frac{\pi}{2})$.

Example 5.2.4. Explain why $\cos(t+2\pi) = \cos(t)$ and $\sin(t+2\pi) = \sin(t)$ for any real number t.

Example 5.2.5. Describe what happens to the values $\cos(t)$ as t increases from 0 to 2π.

5.3 Graphing $\cos(t)$ and $\sin(t)$





Some important things to notice for both $y = \cos(t)$ and $y = \sin(t)$:

- The graphs repeat themselves at regular intervals: they are *periodic*. The gap between successive repetitions is 2π, because $\cos(t+2\pi) = \cos(t)$, and similarly $\sin(t+2\pi) = \sin(t)$. We say that the *period* of these functions is 2π.
- The maximum value (that is, the maximum y-value) is 1.
- The minimum value (that is, the minimum y-value) is -1.
- The *mid-line* of each curve is the t-axis. Since the maximum and minimum values are both distance 1 from this line, we say that the *amplitude* is 1.

Warning. maximum value $-$ minimum value $=$ *double* the amplitude.

5.4 Application: modelling periodic phenomena

Suppose that $f(t)$ is the value of some periodically changing quantity at time t. The *period* of f is the time taken for one cycle to complete.

A simple model of a periodically changing quantity $f(t)$ can often by found by using the following theorem.

Theorem 5.4.1 (Periodic models)**.** *Let a, b, k be constants with $a, b > 0$. If*

$$y = k + a\sin(bt) \quad \textit{or} \quad y = k + a\cos(bt), \quad \textit{then}$$

- *the mid-line is $y = k$, so the average value of y is k,*
- *the amplitude is a,*
- *the maximum value is $k + a$ and the minimum value is $k - a$, and*
- *$b = 2\pi/p$ where p is the period*

For $y = k + a\cos(bt)$, the maximum value is at $t = 0$. For $y = k + a\sin(bt)$, when $t = 0$ the graph it at its mid-line, and it is increasing.

Example 5.4.2. Find the equation of a periodic function $y = y(t)$ with $y(0) = 4$, period 6, amplitude 3 and maximum value 4. Sketch the graph $y = y(t)$.

Example 5.4.3 (Adapted from the textbook, Exercise 1.5.53)**.**
A man breathes once every 5 seconds. His average lung volume is 2500 mL and the volume of air inhaled and exhaled with each breath is 500 mL. Let $V(t)$ be the volume of his lungs in mL at time t (in seconds).

(a) Suggest a plausible model for $V(t)$, given that his lungs are at their average capacity and he is inhaling when $t = 0$.

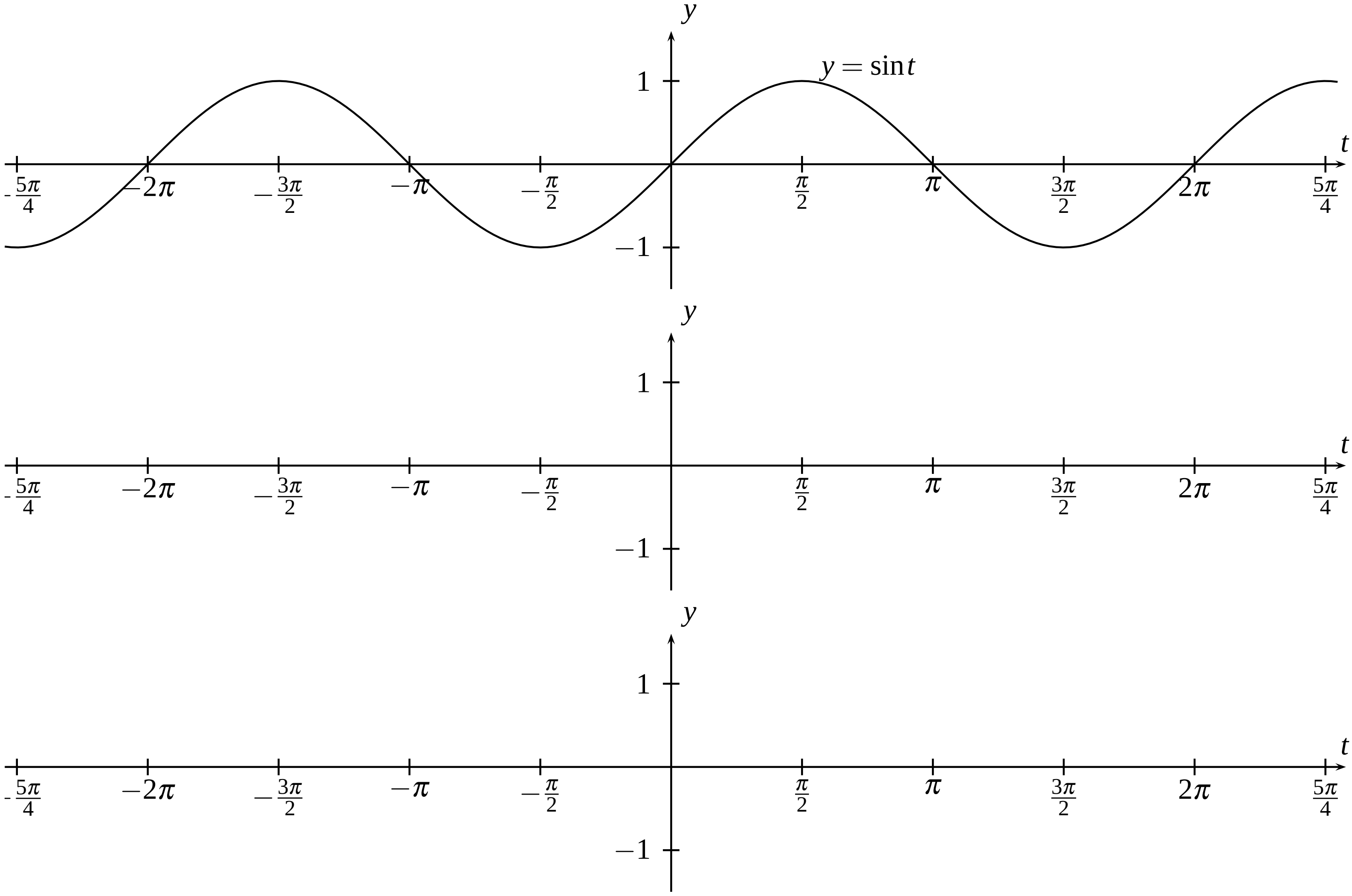
(b) Sketch the graph of $V(t)$.

(c) According to your graph, when are his lungs at their maximum capacity?

5.5 The derivatives of trigonometric functions

Textbook: Section 2.5

Let's estimate the derivatives of sin and cos graphically:



Theorem 5.5.1 (The derivatives of sin and cos)**.**

$$\frac{d}{dx}\big(\sin(x)\big) = \qquad\qquad \textit{and} \qquad \frac{d}{dx}\big(\cos(x)\big) =$$

Warning. These formulae only work if x is measured in radians. In fact, the reason for using radians in the first place is to get nice formulae like these!

Example 5.5.2. Differentiate: (a) $\dfrac{\sin(x)}{2} + 3\cos(x)$

(b) $3 - x - \sqrt{2}\cos(x)$.

The chain rule for trigonometric functions

Textbook: Section 2.8

We have an analogue of the extended power rule we saw before:

Theorem 5.5.3. *If u is a function of x, then*

$$\frac{d}{dx}\big(\cos(u)\big) = -\sin(u)\cdot\frac{du}{dx} \quad \textit{and} \quad \frac{d}{dx}\big(\sin(u)\big) = \cos(u)\cdot\frac{du}{dx}.$$

Example 5.5.4. Compute the derivative of $f(x) = 3+7\cos(2\pi x)$.

Example 5.5.5. What is $\dfrac{d}{dt}\big(\sin(\pi - t^2)\big)$?

5.6 Integrals of trigonometric functions

If we reverse Theorem 5.5.1 then we get:

Theorem 5.6.1.

$$\int \sin(x)\,dx = -\cos(x)+C \quad \textit{and} \quad \int \cos(x)\,dx = \sin(x)+C.$$

Similarly, by reversing Theorem 5.5.3 we get:

Theorem 5.6.2. *If u is a function of x, then*

$$\int \sin(u)\cdot\frac{du}{dx}\,dx = -\cos(u)+C \quad \textit{and} \quad \int \cos(u)\cdot\frac{du}{dx}\,dx = \sin(u)+C.$$

Example 5.6.3. What is the average value of

$$f(t) = 1+2\sin\left(\frac{\pi t}{6}\right)$$

as t varies from 0 to 2?