4 Roots and fractional powers

4.1 Square roots and nth roots

Square roots

Suppose that a is a real number with $a \geq 0$, and consider the equation

$$x^2 = a.$$

This has two real solutions. One is ≥ 0 and the other is ≤ 0.

For example, $x^2 = 9$ has the solutions

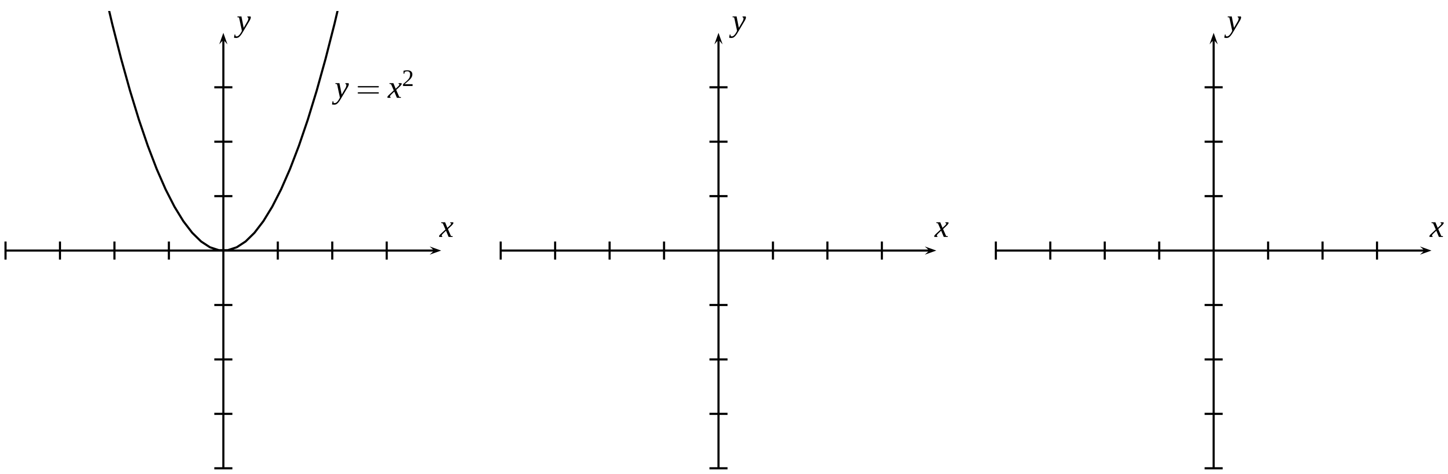
Definition 4.1.1. $\sqrt{a}$ is defined to be non-negative solution to $x^2 = a$.

So, using this notation, the solutions to $x^2 = a$ are $x = \sqrt{a}$ and $x = -\sqrt{a}$. [This is sometimes abbreviated as $x = \pm\sqrt{a}$.]

Definition 4.1.2. The *square root function* is given by $y = \sqrt{x}$. It is defined for $x \geq 0$ and is undefined for $x < 0$.

Note that if $y = \sqrt{x}$ then y is the non-negative number with $y^2 = x$.

Example 4.1.3. Given the graph of $y = x^2$ on the left, swap x and y to get the graph of $x = y^2$, and then take just the non-negative y-values to get the graph of $y = \sqrt{x}$.



Example 4.1.4. For which real numbers x is the function $f(x) = 2 + \frac{1}{2}\sqrt{3-x}$ defined? What are $f(6)$ and $f(-6)$?

nth roots

Now let n be a positive integer, let a be a real number and consider the equation

$$x^n = a.$$

If n is even and $a \geq 0$ then this has two solutions (just like the case $n = 2$), one of which is ≥ 0 and the other is ≤ 0.

If n is odd, then this equation has one solution for any real number a.

Example 4.1.5. Solve the equations $x^4 = 16$ and $x^3 = 27$.

Definition 4.1.6.

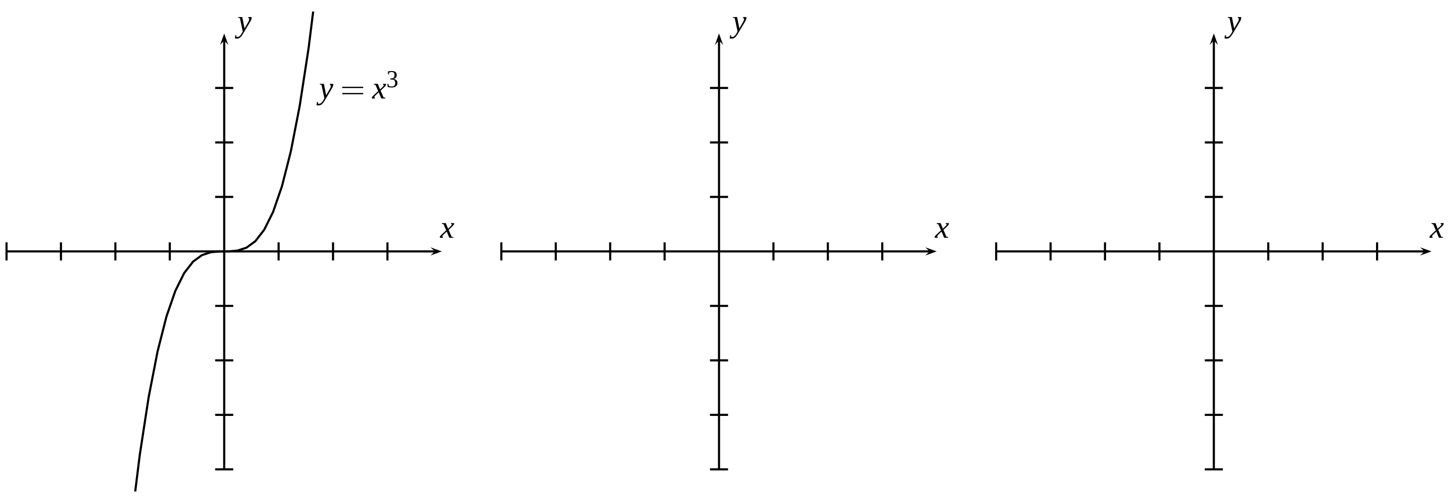
- If n is even and $a \geq 0$ then $\sqrt[n]{a}$ is the non-negative solution to $x^n = a$.
- If n is odd and a is any real number then $\sqrt[n]{a}$ is the solution to $x^n = a$.

For example, $\sqrt[2]{a} =$

For each positive integer n, this defines a function $y = \sqrt[n]{x}$. If n is even then this function is undefined for the negative real numbers x. If n is odd then this function is defined for all real numbers x.

Note that if $y = \sqrt[n]{x}$ then y satisfies $y^n = x$.

Example 4.1.7. Given the graph of $y = x^3$ on the left, swap x and y to get the graphs of $x = y^3$ and $y = \sqrt[3]{x}$



4.2 Fractional powers

Recall that if n is an integer, we defined a^n on page 6:

- $a^0 = 1$; and
- if $n > 0$ then $a^n = a \times \cdots \times a$ is the product of a with itself n times, and $a^{-n} = \dfrac{1}{a^n}$.

Definition 4.2.1. If m, n are positive integers and a is a real number then we define

$$a^{m/n} = \sqrt[n]{a^m} \quad \text{and} \quad a^{-m/n} = \frac{1}{a^{m/n}} = \frac{1}{\sqrt[n]{a^m}}$$

whenever these quantities make sense. [This means that if n is odd then we need $a \geq 0$, and if n is even then a can be any real number].

Example 4.2.2. If $a \geq 0$ then $a^{1/2} = \sqrt[2]{a} = \sqrt{a}$. If $a > 0$ then $a^{-1/2} = \dfrac{1}{\sqrt{a}}$.

In the next statement, the word “fractions” means any fraction like $\frac{5}{6}$, $-\frac{43}{2}$, $10.32 = \frac{1032}{100}$, $2 = \frac{2}{1}$ and so on.

Theorem 4.2.3 (Power laws for fractional exponents)**.**
If a, b are non-negative real numbers and r, s are fractions then:

1. $(ab)^r = a^r \cdot b^r$ *and* $\left(\dfrac{a}{b}\right)^r = \dfrac{a^r}{b^r}$ *if* $b \neq 0$
2. $a^r \cdot a^s = a^{r+s}$ *and* $\dfrac{a^r}{a^s} = a^{r-s}$
3. $(a^r)^s = a^{rs}$

Example 4.2.4. Explain why $(\sqrt[n]{a})^n = a$ and $\sqrt[n]{a^n} = a$ for any $a \geq 0$.

Example 4.2.5. Simplify the following, rewriting them using fractional powers:

(a) $\sqrt[4]{y}$

(b) $\sqrt[3]{8x^2}$

(c) $\sqrt[3]{x^2+1}$

(d) $\dfrac{1}{(\sqrt{25xz^{-2}})^3}$ [Assume that $x > 0$ and $z > 0$].

(e) $\dfrac{y}{\sqrt[5]{y^{10}}}$

Example 4.2.6. Let F denote the femur length, M the mass and W the waist circumference of a healthy adult man. A simple model suggests that M is directly proportional to F^3 and that M is directly proportional to W^2. Show that there is a constant r so that F is directly proportional to W^r, and find the value of r.

4.3 Derivatives of powers

Textbook: Section 2.5

We stated the power rule for derivatives in Theorem 2.2.3, and applied this to positive whole-number powers. In fact, this rule works for any constant power.

Theorem 4.3.1. *The derivative of x^r is rx^{r-1} for any constant power r:*

$$\frac{d}{dx}(x^r) = rx^{r-1}.$$

Example 4.3.2. Find the derivative of $-2x^5 + \frac{7}{\sqrt{x}}$.

Example 4.3.3. If $f(x) = \sqrt[3]{x}$ then $f'(0)$ is not defined. Use the power rule to show that this is true, and draw a picture to explain it.

The extended power rule

Textbook: Section 2.8

Theorem 4.3.4 (The extended power rule)**.** *If u is a function of x, then for any constant power r we have*

$$\frac{d}{dx}\left(u^r\right) = ru^{r-1} \cdot \frac{du}{dx}.$$

Warning. As in the power rule, it is essential that the power r is a *constant*.

Example 4.3.5. Check that this works if $u = 3x + 1$ and $r = 2$.

Example 4.3.6. Find $\frac{dy}{dx}$ if $y = (x^2+1)^5$.

Example 4.3.7. Compute $\frac{d}{dx}\sqrt{(x+6)^3}$.

Example 4.3.8. Compute $\frac{d}{dx}\sqrt{(5x^2+6)^3}$.

Example 4.3.9. Compute $f'(x)$ if $f(x) = 1+x+3\sqrt{(5x^{-3}+6)^3}$.

Example 4.3.10. What is the derivative of $\frac{1}{(2x+3)^{0.3}}$?

4.4 Antiderivatives and integrals of powers

If we reverse Theorem 4.3.1, write $s = r - 1$ and divide by $r + 1$, then we get:

Theorem 4.4.1. *For any constant power s with $s \neq -1$, the antiderivative of x^s is*

$$\int x^s\,dx = \frac{1}{s+1}x^{s+1} + C.$$

Remark 4.4.2. If $s = -1$ then this formula doesn't work, because it tells us to divide by 0. We'll see what the antiderivative of x^{-1} is later on.

Example 4.4.3. What is $\int x^{1/4}\,dx$?

Example 4.4.4. What is $\int \frac{5}{\sqrt{x}}\,dx$?

Example 4.4.5. Compute $\int_1^3 2t^3 - t^{0.8}\,dt$

Example 4.4.6. Find a function $f(x)$ so that $f'(x) = 3\sqrt{x}$ and $f(1) = 3$.

We can also reverse the extended power rule:

Theorem 4.4.7. *If u is a function of x, then for any constant power s we have*

$$\int u^s \cdot \frac{du}{dx}\,dx = \frac{1}{s+1}u^{s+1} + C.$$

Example 4.4.8. What is $\int 2x(x^2-6)^{1/4}\,dx$?

Example 4.4.9. What is $\int x(x^2-6)^{1/4}\,dx$?

Example 4.4.10. What is $\int (3x+1)^{12}\,dx$?

Example 4.4.11. Compute $\int_1^3 \frac{x+1}{\sqrt{x(x+2)}}\,dx$