

## 2 Calculus for polynomials

In this chapter, we'll introduce the derivative, the antiderivative and the definite integral, and we'll see how to compute them for polynomials.

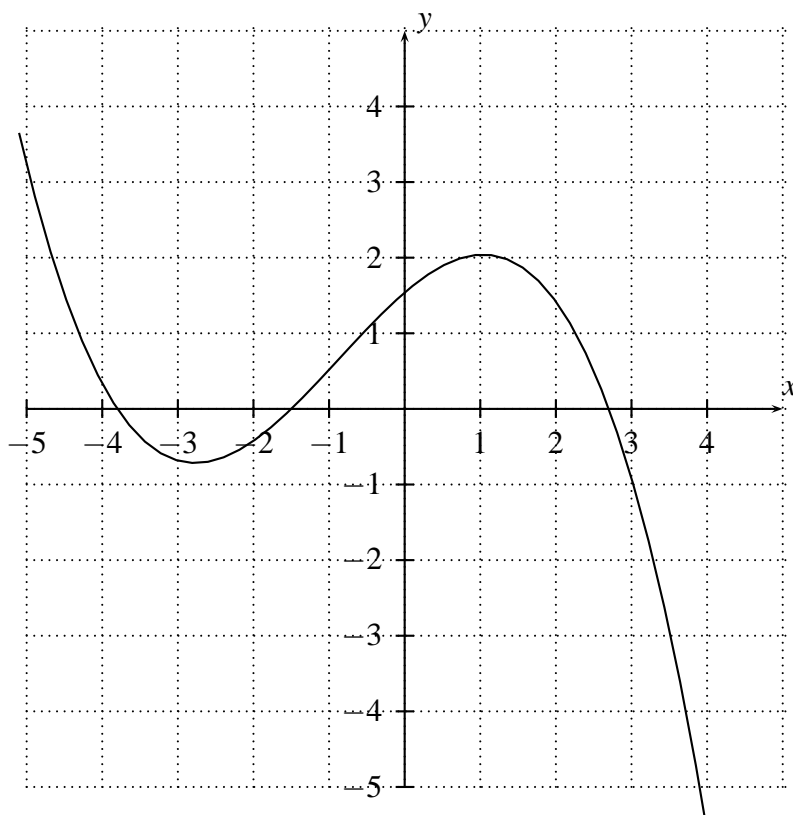
### 2.1 Tangent lines and derivatives

*Textbook: sections 2.4 (pages 101–102) and 2.5 (page 115)*

Let  $f$  be a function and let  $a$  be a real number. A straight line  $L$  is a *tangent line* to  $y = f(x)$  at  $x = a$  if  $L$  meets the graph of  $f$  at  $(a, f(a))$ , and  $L$  points in precisely the same direction as the graph of  $f$  as it passes through  $(a, f(a))$ .

**Example 2.1.1.** The graph of the function  $f(x) = 1.539 + 0.861x - 0.26x^2 - 0.1x^3$  is shown below. Sketch the tangent lines to  $y = f(x)$  at the following  $x$ -values.

- (i)  $x = -3$       (ii)  $x = 0$       (iii)  $x = 2$ .



#### Why is the idea of a tangent line important for scientists?

We'll see that the slope of a tangent line to the graph of a function tells you how rapidly the function is changing. For example, if the function models the number of blood cells in a patient's bloodstream at a given time, then measuring the slopes of the tangent lines will tell you how rapidly the number of blood cells is increasing or decreasing.

**Definition 2.1.2.** If  $f$  is a function and  $a$  is a real number, then *the derivative of  $f$  at  $x = a$*  is the slope of the tangent line to  $y = f(x)$  at  $x = a$ , provided this makes sense.\*

\* "Provided this makes sense" means that there must be a single unambiguous tangent line to  $y = f(x)$  at  $x = a$ , and it must have a slope, meaning that it cannot be vertical. We won't worry too much about this for most of this course. In fact we won't worry about it at all in this section, because for polynomials, these conditions are always true.

For each value of  $a$ , the derivative of  $f$  at  $x = a$  is a real number (since it's the slope of a line). We write it as  $f'(a)$ .

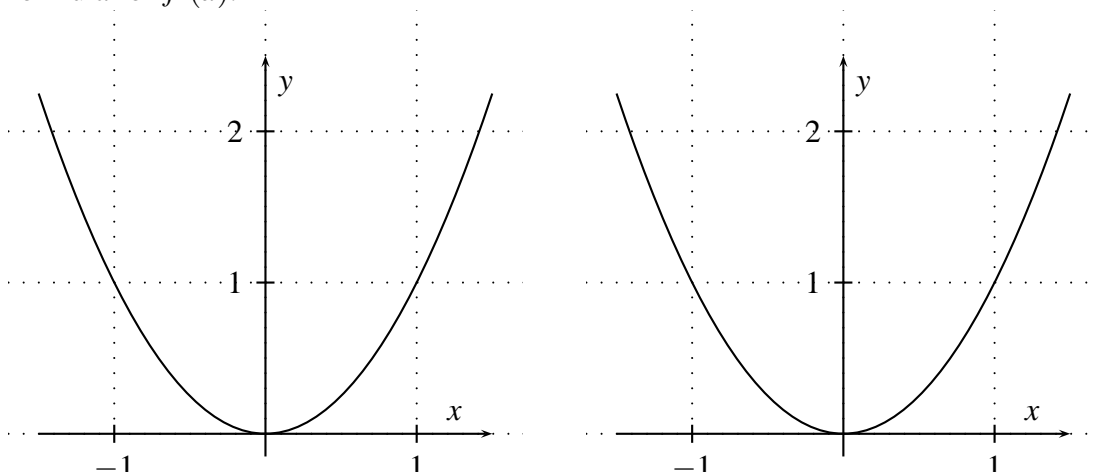
**Notation.** There are several different ways to write the derivative  $f'(a)$ :

$$f'(a), \quad \frac{df}{dx}(a), \quad \left. \frac{df}{dx} \right|_{x=a}, \quad \text{and if } y = f(x), \text{ we can write } \frac{dy}{dx}(a) \text{ or } \left. \frac{dy}{dx} \right|_{x=a}.$$

They all mean exactly the same: the slope of the tangent line to the graph of  $f$  at  $x = a$ . Sometimes it's more convenient to write the derivative in one way than the other.

**Example 2.1.3.** Let  $f(x) = 1.539 + 0.861x - 0.26x^2 - 0.1x^3$ . Use your answer to Example 2.1.1 and the slope formula in Theorem 1.2.16 to estimate  $f'(0)$  and  $f'(2)$ .

**Example 2.1.4.** Let  $f(x) = x^2$ . Use the graphs of  $y = x^2$  below to estimate  $f'(-1)$ ,  $f'(-\frac{1}{2})$ ,  $f'(0)$ ,  $f'(\frac{1}{2})$  and  $f'(1)$ . Plot your results on a graph and guess a general formula for  $f'(a)$ .



## 2.2 Calculating derivatives of polynomials

*Textbook: section 2.5*

One way to roughly calculate  $f'(a)$  is to draw a graph, draw something approximating the tangent line and estimate its slope. But this is error-prone and time-consuming. Now we'll see how to find  $f'(a)$  exactly, without having to draw graphs.


**Definition 2.2.1.** For each value of  $a$  we have a number  $f'(a)$ . This defines a function  $f'$ . If  $y = f(x)$  then we also write this function as  $\frac{dy}{dx}$ . This function is called the *derivative* of  $f$ .

We also talk about *differentiating* a function  $f(x)$ , which means “finding the derivative of  $f(x)$ ”.

**Notation.** If  $y = f(x)$ , then

$$f'(x) \quad \text{and} \quad \frac{df}{dx} \quad \text{and} \quad \frac{d}{dx}(f(x)) \quad \text{and} \quad \frac{dy}{dx}$$

all mean exactly the same function: the derivative of  $f$ . Sometimes it's more convenient to write the derivative in one way than the other.

 **Warning.** Although  $\frac{d}{dx}$  looks like a fraction, it is not, it is special notation meaning “differentiate what follows”. Make sure you don't treat it as a fraction. For example, the following attempt at cancellation is *wrong*:

$$\frac{d}{dx}(x^2) = \frac{dx^2}{dx} = \frac{\cancel{dx}^2}{\cancel{dx}} = x$$

We can use the following rules to differentiate any polynomial.

**Theorem 2.2.2.** If  $k$  is a constant, then  $\frac{d}{dx}(k) = 0$ .

**Theorem 2.2.3.** The derivative of  $x^n$  is  $nx^{n-1}$ :

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

**Theorem 2.2.4** (Multiplying by constants and differentiation).

If  $k$  is a constant, then

$$\frac{d}{dx}(kf(x)) = k \frac{d}{dx}(f(x)).$$

**Theorem 2.2.5** (The sum-difference rule for derivatives).

- The derivative of a sum is the sum of the derivatives:

$$\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x)).$$

- The derivative of a difference is the difference of the derivatives:

$$\frac{d}{dx}(f(x) - g(x)) = \frac{d}{dx}(f(x)) - \frac{d}{dx}(g(x)).$$

**Example 2.2.6.** If  $f(x) = x^7$ , compute  $f'(x)$ .

**Example 2.2.7.** What is  $\frac{d}{dx}(x^2 + 4)$ ?

**Example 2.2.8.** What is  $\frac{d}{dt}(t^2 + 4)$ ?

**Example 2.2.9.** Compute  $\frac{d}{dx}(\frac{1}{8}x^6 - 3x)\Big|_{x=2}$ . What does this mean?

**Example 2.2.10.** If  $f(x) = -3(2x - 4)^2 - x^2$ , find  $f'(0)$ .

**Example 2.2.11.** If  $f(x) = 1.539 + 0.861x - 0.26x^2 - 0.1x^3$ , what are  $f'(0)$  and  $f'(2)$ ? Compare with Example 2.1.1.

**Example 2.2.12.** If  $y = \frac{3x^4 + 1}{4}$ , what is  $\frac{dy}{dx}$ ?

**Remark 2.2.13.** We can differentiate every polynomial using the rules on the previous page. Later on, we'll discuss more rules which can be used to differentiate functions which aren't polynomials, such as  $f(x) = x\sqrt{x^2 + 1}$ .

## 2.3 Antiderivatives

*Textbook: section 5.1*

In section 0, we saw that it's useful to be able to undo things like addition and multiplication. It's also useful to be able to undo differentiation, or in other words, to solve the following problem.

Suppose that  $F(x)$  is a function, and you don't know its formula but you do know the formula for its derivative  $F'(x)$ . Can you find the original function  $F(x)$ ?

**Example 2.3.1.** Write down three different functions whose derivatives are  $3x$ .

### Why are antiderivatives important for scientists?

In some situations it might be easier to measure the derivative of something that you are interested in rather than measuring it directly. For example, suppose you can measure the speed at which something travels, but not the total distance it travels. Since speed is the rate of change of distance travelled, that is, the derivative of distance travelled, the antiderivative of speed gives you the distance travelled.

In section 2.5, we'll also see that antiderivatives let you calculate definite integrals. These have plenty of applications.

### Definition 2.3.2.

If  $F'(x) = f(x)$  then we say that  $F$  is an *antiderivative* of  $f$ .

**Example 2.3.3.** Write down three different antiderivatives of  $f(x) = 3x$ .

A function  $f$  can have several antiderivatives. But they are all related:

### Theorem 2.3.4.

If  $F(x)$  is an antiderivative of  $f(x)$ , then the most general antiderivative of  $f(x)$  is

$$F(x) + C \quad \text{where } C \text{ is a constant.}$$

In other words, for every constant  $C$ , the function  $F(x) + C$  is an antiderivative of  $f(x)$ , and there are no others.


**Example 2.3.5.** What is the most general antiderivative of  $f(x) = 3x$ ?

**Definition 2.3.6.** If  $f$  is a function then we write

$$\int f(x) dx$$

for the most general antiderivative of  $f(x)$ . We will call  $\int f(x) dx$  the *indefinite integral* of  $f$ .

Because of Theorem 2.3.4, the answer to  $\int f(x) dx$  will always be a formula ending in “ $\cdots + C$ ” where  $C$  is a constant”.

 **Warning.** The symbol  $dx$  in  $\int f(x) dx$  is simply an indication that we should treat  $x$  as the independent variable when we antidifferentiate; we are not multiplying  $f(x)$  by  $dx$  here.<sup>†</sup> This also means that if you have an integral like  $\int 3t dt$  then you have to antidifferentiate using  $t$  as the independent variable instead of  $x$ . In other words, you want to find something so that  $\frac{d}{dt}$  of your answer is  $3t$ .

**Example 2.3.7.** Find  $\int 3x dx$  and  $\int 3t dt$ .

**Example 2.3.8.** Find  $\int 6x^5 dx$ .

**Example 2.3.9.** Find  $\int x^5 dx$ . Explain why the derivative of your answer should be  $x^5$ , and check that this is true.

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<sup>†</sup>This is just the same role that  $dx$  plays in the expression  $\frac{d}{dx}(f(x))$ ; we are not dividing anything by  $dx$ . Instead, it tells us that  $x$  is the independent variable so we should differentiate with respect to  $x$ .

Here are three rules that we can use to find the antiderivative of any polynomial.

**Theorem 2.3.10.** For  $m = 0, 1, 2, 3, \dots$ , the antiderivative of  $x^m$  is

$$\int x^m dx = \frac{1}{m+1} x^{m+1} + C.$$

**Remark 2.3.11.** Two special cases of this rule are easy to miss: if  $m = 0$  then  $x^m = 1$  and  $\frac{1}{m+1} x^{m+1} + C = \frac{1}{0+1} x^{0+1} + C = 1 \cdot x^1 + C = x + C$ , so we get

$$\int 1 dx = x + C$$

and if  $m = 1$  then  $x^m = x$ , so we get

$$\int x dx = \frac{1}{2} x^2 + C.$$

**Theorem 2.3.12** (Multiplying by constants in antiderivatives). *If  $k$  is a constant, then*

$$\int k f(x) dx = k \int f(x) dx.$$

**Theorem 2.3.13** (The sum-difference rule for antiderivatives).

$$\begin{aligned} \int f(x) + g(x) dx &= \int f(x) dx + \int g(x) dx \\ \int f(x) - g(x) dx &= \int f(x) dx - \int g(x) dx \end{aligned}$$

**Example 2.3.14.** Compute the general antiderivative of  $x^7$ , and check that your answer is correct.

**Example 2.3.15.** What is  $\int 4 dx$ ?

**Example 2.3.16.** What is  $\int x^2 + 4 dx$ ? What is  $\int t^2 + 4 dt$ ?

**Example 2.3.17.** If  $f(t) = \frac{1}{8}t^6 - 3t$ , find  $\int f(t) dt$ . What does this mean? Check that your answer is correct.

**Example 2.3.18.** Find  $\int -3(2x - 4)^2 - x^2 dx$ .

**Example 2.3.19.** Compute  $\int \frac{3x^4 + 1}{4} dx$ .

**Example 2.3.20** (Textbook, Exercise 5.4.36). Alice and Bob are asked to memorise words for 10 minutes. After  $t$  minutes have passed, suppose that Alice has memorised  $a(t)$  words and Bob has memorised  $b(t)$  words. The rate at which they memorise words found to be

$$\text{Alice: } a'(t) = -0.009t^2 + 0.2t \text{ words per minute,}$$

$$\text{Bob: } b'(t) = -0.003t^2 + 0.2t \text{ words per minute.}$$

Who has the higher rate of memorisation, and how many more words does that person memorise over the 10 minute experiment?

**Remark 2.3.21.** We've seen three rules for antiderivatives above, which are analogues of the ones we looked at for derivatives. These are enough to allow us to find the antiderivative of any polynomial. There are more rules, but we'll save these until we look at more complicated antiderivatives later on.



## 2.4 Areas and definite integrals

*Textbook: sections 5.2*

Suppose that  $R$  is a region in the  $x$ - $y$  plane. Split  $R$  into two pieces:

$R^+$  = the part of  $R$  above the  $x$ -axis, and

$R^-$  = the part of  $R$  below the  $x$ -axis.

Let  $A^+$  be the area of  $R^+$  and  $A^-$  be the area of  $R^-$ . The *net signed area* of  $R$  is given by  $A = A^+ - A^-$ .

**Definition 2.4.1.** The *definite integral* of a function  $f(x)$  from  $a$  to  $b$  is the net signed area of the region bounded by the  $x$ -axis, the graph  $y = f(x)$  and the vertical lines  $x = a$  and  $x = b$ . This number is written as

$$\int_a^b f(x) dx.$$

**Example 2.4.2.** Sketch  $y = 3x$ , and hence find  $\int_0^2 3x dx$  and  $\int_{-2}^2 3x dx$ .

## 2.5 The fundamental theorem of calculus

*Textbook: section 5.3*

**Theorem 2.5.1** (The first fundamental theorem of calculus, FTC1).

*Let  $f$  be a function, let  $a$  be a fixed number, and let*

$$A(x) = \int_a^x f(t) dt.$$

*Then  $A'(x) = f(x)$ .*

**Theorem 2.5.2** (The second fundamental theorem of calculus, FTC2).

If  $F$  is an antiderivative of  $f$  then

$$\int_a^b f(x) dx = F(b) - F(a).$$

The FTC2 establishes a link between antiderivatives (that is, indefinite integrals) and definite integrals, and it is the main tool used to compute definite integrals.

We often write

$$\left[ F(x) \right]_a^b \quad \text{or} \quad F(x) \Big|_a^b \quad \text{as a shorthand for} \quad F(b) - F(a)$$

when computing definite integrals.

**Example 2.5.3.** Use the FTC2 to find  $\int_0^2 3x dx$ .

**Example 2.5.4.** What is  $\int_{-1}^2 x - 1 dx$ ? Use the FTC2 to find the answer, and then draw a sketch to check it.

In Theorems 2.3.12 and 2.3.13 we saw how indefinite integrals behave when we multiply by constants, add or subtract them. Because the FTC2 says that antiderivatives and definite integrals are very closely linked, we have similar rules for definite integrals:

**Theorem 2.5.5** (Multiplying by constants in definite integrals). *If  $k$  is a constant, then*

$$\int_a^b kf(x) dx = k \int_a^b f(x) dx.$$

**Theorem 2.5.6** (The sum-difference rule for definite integrals).

$$\int_a^b (f(x) + g(x)) dx = \int_a^b f(x) dx + \int_a^b g(x) dx$$

$$\int_a^b (f(x) - g(x)) dx = \int_a^b f(x) dx - \int_a^b g(x) dx$$

**Example 2.5.7.** Compute  $\int_0^2 x^5 - 2x^2 dx$ .

**Example 2.5.8.** Calculate  $\int_{-1}^1 \frac{3(x+1)^2 - x^9}{2} dx$