

1 Functions, graphs and polynomials

1.1 Coordinates, graphs and functions

Textbook: Appendix B

What is a function?

Definition 1.1.1. A *function* f is a rule with a collection of possible input values so that for each input value x there is exactly one output value, $f(x)$.



Warning. Remember that $f(x)$ does *not* mean “ f multiplied by x ” since this doesn’t make sense (you can’t multiply by a function). Instead, $f(x)$ means the output value of the function f when you feed in the input value x .

Remark 1.1.2. Some functions are not defined for every real number. For example, $f(x) = \sqrt{x}$ is only defined for $x \geq 0$. It’s often important to keep track of the x -values for which a function is defined.

Why are functions important for scientists?

One of the key tasks of science is to formulate numeric predictions about the way things behave. Functions allow us to express these predictions mathematically, and to make further predictions. Functions which are used in this way to represent quantitative scientific predictions are called *models*.

For example, suppose that a man sleeps in an area with mosquitos. We might model the relationship between the area of his exposed skin and the number of mosquito bites he receives in a night using the function

$$f(x) = 300x$$

where $f(x)$ is the number of mosquito bites and x is the area of exposed skin in m^2 . If we know that the man sleeps with 0.3m^2 of exposed skin (the input value for our function/prediction), then this model predicts that he will be bitten by mosquitos $f(0.3) = 300 \times 0.3 = 90$ times (the output value).

In this example, $f(x)$ is defined for $x \geq 0$ and $x \leq A$ where A is the man’s total skin surface area.

In this course, we will be concerned with functions where the input values and output values are real numbers. Often, a function will be specified by giving a formula relating the input value (often written x , or t) to the output value (often written y , or $f(x)$, $g(t)$, etc). For example,

$$f(x) = 1 + x \quad \text{where } x \text{ is a real number with } x \geq 0$$

specifies a function which is defined for $x \geq 0$.

If we are given a formula for a function $f(x)$ but are not told where the function is defined, we generally assume that it’s defined for all the real numbers where the formula makes sense (and defines a real output number).

Example 1.1.3. A function f is defined by the rule $f(x) = 2 - x$.

(a) For which real numbers is $f(x)$ defined?

(b) Find $f(3)$ and $f(-3)$.

(c) Simplify $f(x + 4)$.

Example 1.1.4. A function g is given by $g(x) = \frac{x - 4}{x(x - 3)}$.

(a) For which real numbers is $g(x)$ defined?

(b) Find $g(3)$ and $g(-3)$.

(c) Simplify $g(x + 4)$.

It is also useful to be able to write down rules defining functions which involve more than one formula. Here is an example.

Example 1.1.5. If we write

$$f(x) = \begin{cases} 2x & \text{if } x \leq 2, \\ 6-x & \text{if } x > 2 \end{cases}$$

then we are defining a single function, $f(x)$, “in two pieces”. For example,
 if $x = 1$ then $x \leq 2$, so $f(1) =$ and
 if $x = 4$ then $x > 2$, so $f(4) =$

For which real numbers is $f(x)$ defined?

Coordinates and graphs

Each point in the plane has a pair of coordinates (x,y) where x and y are real numbers.

Given a function f , we can illustrate f by plotting all of the points (x,y) where $y = f(x)$ and $f(x)$ is defined. This is called the *graph* of the function f , or the graph of the equation $y = f(x)$.

Why are graphs important for scientists?

If we know the graph of a model function f then we can easily describe how the model behaves when we change the input values; for example, we can say whether increasing the input value will increase or decrease the output value.

We can also use the graph to look up values of the function. So knowing the graph of a function is enough to use it to make simple predictions.

Example 1.1.6. The number of mosquito bites a man receives in a night is modelled by $f(x) = 300x$ where x is the surface area of the man's exposed skin. Plot the graph of $f(x)$ for $0 \leq x \leq 2$, and mark the point $(0.3, f(0.3))$ on your graph. Does $f(x)$ increase or decrease as x increases?

Example 1.1.7. Plot the graphs of the functions

$$f(x) = \begin{cases} 2x & \text{if } x \leq 2, \\ 6-x & \text{if } x > 2 \end{cases} \quad \text{and} \quad g(x) = x(4-x)$$

on the same diagram. Hence find the values of x with $f(x) > g(x)$.

1.2 Slope and linear functions

Textbook: section 1.1

Linear functions are of the form $f(x) = mx + b$ where m and b are constants (which just means that they represent fixed numbers, and don't depend on x). For example, these are linear functions:

$$f(x) = 2x + 1, \quad g(x) = -x + 4, \quad h(x) = x, \quad k(x) = 7$$

but $f(x) = x^2 + 1$ is not linear.

Theorem 1.2.1. *The graph of any linear function is a straight line. So to plot the graph of a linear function, you only need to know two points on the graph, and then you can join them with a ruler and extend the line in both directions.*

Why are linear functions important for scientists?

Linear functions are important because they are used in many simple models. For example, when you draw a line of best fit through a plot of experimental data then you are constructing a linear model: the linear function $f(x)$ whose graph is the straight line you drew. You can then use this model to make predictions.

Horizontal lines and constant functions

Example 1.2.2. Plot the graphs of $f(x) = 3$, $g(x) = 0$ and $h(x) = -1.5$.

Functions of the form

$$f(x) = b \quad \text{where } b \text{ is a constant}$$

(meaning that b is a fixed real number and doesn't depend on x) are called *constant functions*. Their output values never change, no matter which input value x is chosen. The graph of $f(x) = b$ is the horizontal line passing through $(b, 0)$.

Lines through the origin and direct proportion

Theorem 1.2.3 (Lines through the origin). *Let m be a constant.*

The graph of $y = mx$ (or $f(x) = mx$) is the straight line through the points $(0,0)$ and $(1,m)$.

Definition 1.2.4. The *slope* of the line $y = mx$ is m .

Example 1.2.5. Sketch the graphs of $y = 1.3x$ and $y = -0.3x$. What are the slopes of these lines?

Definition 1.2.6. We say that y is *directly proportional* to x if $y = mx$ for some constant $m > 0$.

Example 1.2.7. It is found experimentally that the average heart mass of an adult mammal is directly proportional to the mass of the mammal. Given that a typical 80 kg human's heart weighs 320 g, estimate the heart mass of a 200 g rat.

Linear functions $f(x) = mx + b$

Theorem 1.2.8 (The graph of $y = mx + b$). *Let m and b be constants.*

The graph of $y = mx + b$ is the straight line parallel to $y = mx$ which passes through $(0, b)$. In other words, it is the straight line through $(0, b)$ and $(1, m + b)$.

Example 1.2.9. Draw the graph of $y = -0.5x + 2$.

Definition 1.2.10. A *linear function* is a function of the form $f(x) = mx + b$ where m and b are constants.

The *slope* of the line $y = mx + b$ is m , and the *y-intercept* of the line is the point $(0, b)$.

The equation $y = mx + b$ is called the *slope-intercept* equation of the line.

Example 1.2.11. The slope of a horizontal line $y = b$ is zero.

Example 1.2.12. Find the slope-intercept equation of the line with slope 2 passing through the point $(-3, 4)$.

Theorem 1.2.13 (The point-slope equation of a line). *A line with slope m which passes through a given point (x_1, y_1) has equation*

$$y - y_1 = m(x - x_1).$$

This is called the point-slope equation of the line.

Example 1.2.14. Find the point-slope equation of the line with slope 2 passing through the point $(-3, 4)$. Check that it's equivalent to the slope-intercept equation of the same line that we found earlier!

Remark 1.2.15. If you have to find the equation of a line then it usually doesn't make any difference whether you give the slope-intercept equation or the point-slope equation, because they're always equivalent.

Theorem 1.2.16 (Computing the slope from two points). *If (x_1, y_1) and (x_2, y_2) are two points on a line, then the slope of the line is given by*

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

Example 1.2.17. Find the equation of the line passing through $(1, 0)$ and $(-4, 7)$.

1.3 Quadratic functions

Textbook: section 1.2

A quadratic function is a function of the form $f(x) = ax^2 + bx + c$ for some constants a, b, c with $a \neq 0$. Here are some examples of quadratic functions:

$$f(x) = 4x^2 + 3x + 2, \quad g(x) = -x^2 + 4x, \quad h(x) = \frac{x^2 - 3}{4}, \quad q(x) = -0.2x^2$$

The shape of the graph $y = ax^2 + bx + c$ is called a parabola. For example, the shape traced out by a cannon ball as it travels through the air is part of a parabola which opens downwards. A parabola always has a vertical line of mirror symmetry.

Remark 1.3.1. The parabola opens upwards if $a > 0$ and downwards if $a < 0$.

Example 1.3.2. Sketch the graph of $f(x) = -x^2 + 3x + 4$.

Solving $ax^2 + bx + c = 0$

Solving an equation involving a variable x means finding every possible value of x for which the equation is true. If the equation is equivalent to $f(x) = 0$ where f is a quadratic function then there are two ways to attack the problem: factorisation or by using the quadratic formula. We'll just look at the quadratic formula.

Theorem 1.3.3 (The quadratic formula). *Let a, b, c be constants with $a \neq 0$. If $b^2 - 4ac \geq 0$ then the solutions of $ax^2 + bx + c = 0$ are given by*

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

[This means that there are two solutions: $x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$].

If $b^2 - 4ac < 0$ then no real numbers x are solutions.

Example 1.3.4. A volcanic bomb is ejected from a volcano protruding from an ocean. While in the air, its trajectory satisfies $h(x) = 240 + 0.5x - 0.002x^2$ where $h(x)$ is the height of the bomb above sea level and x is the distance of the bomb from the volcano, both measured in metres.

- (a) How high above sea level does the bomb leave the volcano?
- (b) How far from the volcano does the bomb hit the sea?

1.4 Other polynomials

Definition 1.4.1. A *polynomial* is a function $f(x)$ which can be written using only the following ingredients:

1. the variable x ,
2. positive integer powers, and
3. addition, subtraction and multiplication by constants, and division by non-zero constants.

For example, the following expressions all define polynomials:

- $f(x) = x^4$
- $f(x) = 3$
- $f(x) = \frac{2x^6-3}{5}$
- $f(x) = \frac{1}{2}(x-1)^5 + x^5 - 9x^3 - 1.2x$
- constant functions like $f(x) = 2$ and $f(x) = -3.392$
- linear functions like $f(x) = x + 5$ and $f(x) = -4x$
- quadratic functions like $f(x) = -4x^2 + 2x - 1$ and $f(x) = (x-3)^2$
- cubic functions like $f(x) = 2x^3 - 0.4x^2 + x + 1$

Why are polynomials important for scientists?

Polynomials are good functions to consider when trying to find a model. For one thing, there are lots of polynomials, and they can be used to model a wide variety of situations (not just linear relationships or quadratic relationships). Also, they are relatively easy to handle mathematically.

Remark 1.4.2. If $f(x)$ is a polynomial, then $f(x)$ is defined for every real number x . The point is that positive integer powers of any real number, the sum, difference and product of any real numbers are all defined. addition, subtraction and multiplication by constants or dividing by non-zero constants never causes any problems.

However, $f(x) = \sqrt{x}$ is an example of a function which is not a polynomial, since it can't be written using only positive integer powers of x and constants. Another way to see this is to notice that it's not defined for $x < 0$, so it can't be a polynomial because of the last remark.

Example 1.4.3. Is $f(x) = \frac{x+1}{4}$ a polynomial? Is $g(x) = \frac{4}{x+1}$?