

1M01 Mathematical Methods 2010–2011

Lecturer Rupert Levene, email: levene@maths.tcd.ie, tel: 01 896 3945

Office Room 2.9, maths department, Hamilton Building

Course website <http://www.maths.tcd.ie/~levene/1M01>

Recommended textbook *Calculus for the life sciences* by Marvin Bittinger, Neal Brand and John Quintanilla, Pearson 2005.

Timetable Three calculus lectures per week:

Mondays 16:00–16:50, Wednesdays 12:00–12:50 and Thursdays 9:00–9:50.

These are all in the

Joly Theatre, Hamilton Building (Theatre 4).

You'll also have one calculus tutorial per week. This may be in your online timetable already, but tutorials haven't started yet. I'll tell you when they begin.

Assessment Final grade: 50% Calculus (including 10% calculus tutorial work) and 50% Discrete Mathematics

Syllabus

1. Functions and graphs. Lines, polynomials, rational functions, trigonometric functions and the unit circle.
2. Differentiation. Average rate of change, first principles definition, basic rules for differentiation.
3. Graphical interpretation of derivatives, finding maxima and minima.
4. Exponential and log functions. Growth and decay applications.
5. Integration (definite and indefinite). Techniques of substitution and integration by parts. Applications.
6. Differential equations and initial value problems, solving first order linear equations. Applications.

About these notes

The purpose of these outline notes is to reduce the amount of writing we have to do during lectures. However, they are far from complete! You will need to take down the solutions to examples, illustrations and other material from the lectures as we go along.

Course overview

Mathematics is the language in which much of science is expressed. Experimental results are almost always numerical, and mathematics can help us to make sense of them. Typically, these experimental results are used to try to find a model, which for our purposes is a mathematical function which expresses a relationship between various quantities. The model can then be analysed to make predictions about the real world.

Finding a model from experimental results uses parts of mathematics such as statistics, which we won't discuss in these lectures. Instead, we'll focus on analysing models: taking a model and trying to figure out some of its key features. The mathematics we'll use to do this is called calculus, which is the study of how functions change. Using calculus, we be able to answer a variety of questions about models.

For example, suppose you have a model (a function) which expresses how the hatching time for a duck egg depends on the temperature. To use such a model to draw some conclusion, such as finding the temperature which minimizes the hatching time, requires some understanding of the model and the underlying mathematics. It turns out that good way to tackle this particular problem is to use the derivative, which tells you how rapidly a function is changing.

The derivative is one of the two main tools used in calculus, and the other is the integral. We'll begin the course by reviewing some algebra and discussing graphs, and then we'll be ready to introduce the derivative and the integral for simple sorts of functions. We'll see some applications and then introduce some more intricate functions and their calculus, and see that these have important applications of their own.

0 Algebra review

Textbook: Appendix A

We begin by reviewing some terminology and rules for algebraic manipulation.

- A **real number** is a number such as -2 , -0.2 , 0 , $\frac{1}{2}$, π , $\frac{23}{3}$, 10 . Real numbers are useful for representing quantities that can be measured, such as temperatures measured in $^{\circ}\text{C}$.

A real number a is:

- *positive* if it is greater than than zero, written $a > 0$;
 - *negative* if it is less than than zero, written $a < 0$;
 - *non-negative* if it is zero or positive, written $a \geq 0$;
 - *non-positive* if it is zero of negative, written $a \leq 0$.
 - *non-zero* if it is not 0;
 - an *integer* if it is a whole number such as -3 , 0 or 9 .
- A **variable** is a letter such as $a, b, A, B, x, y, \alpha, \beta$ which represents a real number. Anything you can do to variables, you can also do to numbers, by replacing the variables with numbers (this is called **substitution**). By understanding the relationships between expressions involving variables we can gain an understanding of numerical relationships.

- An **operation** is a rule for combining some numbers or variables to get another number or variable. For example, addition is an operation, represented by “+”, which combines x and y into their sum $x + y$.
- A **function** is a rule assigning one real number to another (more on this soon). For example, if $f(x) = x + 3$ then f is a function which assigns 0 to 3, -2 to 1, and so on. We say that “the value of f at 0 is 3”, and write $f(0) = 3$, $f(-2) = 1$, $f(x + y) =$, etc.
- **Brackets** are matching pairs of symbols such as $()$, $[\]$ etc. They are used in two different ways.

The first way we have just seen: an expression such as $f(a + 1)$ where f is a function means “the value of the function f at $a + 1$ ”.

The second way is to clarify the order in which operations are to be done. The rule is: do the operations in the brackets before the operations outside. For example, the value of $2 \times (4 - 3)$ is but the value of $(2 \times 4) - 3$ is

- An **expression** is a string of numbers, variables, operations, functions, brackets (and so on) which, by substituting certain real numbers for variables, we can interpret as a real number. For example, $x + 2y + 3$ is an expression, since if we set x and y to be any numbers then we know how to compute its value. Another example: \sqrt{x} is an expression, since if we set x to be any real number which is not negative then we can compute its value.

Two expressions are **equal** if they have the same values no matter which numbers are substituted for their variables.

0.1 Arithmetic operations

These are: addition, subtraction, multiplication, division and negation.

Addition To add two expressions, we write them next to one another with a + in between. The result is called the **sum** of the two expressions. Adding 0 leaves an expression unchanged.

Subtraction To subtract two expressions, we write them next to one another with a – in between. The result is called the **difference** of the two expressions. Subtracting 0 leaves an expression unchanged.

Since order does matter when we subtract, that is, $x - y$ and $y - x$ are different, we often need to put brackets around the second expression. For example, to subtract $y + z$ from x we should write $x - (y + z)$, since this is different to $x - y + z$.

Addition and subtraction are related by the rules:

$$(a + b) - b = a \quad \text{and} \quad (a - b) + b = a.$$

This means that we can always “undo” addition by subtraction, and we can undo subtraction by addition.

Multiplication To multiply two expressions, we write them next to one another, using brackets if needed. The result is called the **product** of the two expressions. For example, the product of 2 and x is $2x$, and the product of 2 and $x + 1$ is $2(x + 1)$.



Warning. When we multiply expressions which contain operations, we usually need to use brackets. For example, the brackets in $2(x + 1)$ are needed, since if we omit them then we get $2x + 1$, which is different. See page 7 for more on this.

If it makes things clearer, we sometimes put a dot \cdot or a multiplication sign \times to indicate multiplication. For example, the product of 2 and 3 could be written as 23, but it's much clearer to write $2 \cdot 3$ or 2×3 .

Multiplying by 1 leaves an expression unchanged, and the product of anything with 0 is 0. In fact, we can say more about zero:

Theorem 0.1.1 (The zero principle). *If a product of expressions is zero, then one of the expressions is zero.*

For example, if you know that $a(b + 4) = 0$ then

Division If we are given two expressions then we can divide the first by the second, provided the second is non-zero. The result is called the **quotient** of the two expressions. This is written by separating the expressions with $/$ or \div , or more usually by stacking them into a fraction such as $\frac{2}{3}$ or $\frac{x+y}{a+b}$. Dividing by 1 leaves an expression unchanged.

We cannot divide by zero. This means that an expression like $\frac{a}{b}$ is only defined for $b \neq 0$.

Multiplication and division are related by the rules:

$$\frac{(ab)}{b} = a \quad \text{and} \quad \left(\frac{a}{b}\right) \cdot b = a \quad \text{if } b \neq 0.$$

This means that we can “undo” multiplication by a *non-zero* quantity with division, and we can undo division by a non-zero quantity with multiplication.

Negation The negation of an expression is the product of -1 with that expression. The negation of a is written as $-a$; often a should be enclosed in brackets since, for example, the negation of $x + y$ is $-(x + y)$, which is not the same as $-x + y$.

Negation undoes itself, since $-(-a) = a$.

0.2 Manipulation of arithmetic operations

Basic rules If a, b and c are variables representing real numbers or algebraic expressions, then


- $a + b = b + a$ and $ab = ba$
- $(a + b) + c = a + (b + c)$ and $a(bc) = (ab)c$
- $a - a = a + (-a) = 0$
- $-(b + c) = -b - c$
- $a(b + c) = ab + ac$ and $a(b - c) = ab - ac$ (the Distributive Law).

Example 0.2.1. Simplify $2a - 5a$.

Example 0.2.2. Expand $(x + y)(x - y)$ using the Distributive Law.

Fractions We have:

- $\frac{a}{a} = 1$ and $\frac{a}{1} = a$
- $\frac{a}{c} \cdot \frac{b}{d} = \frac{ab}{cd}$
- $a \cdot \frac{b}{c} = \frac{a}{1} \cdot \frac{b}{c} = \frac{ab}{c}$
- $\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$ and $\frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$
- $\frac{a}{c} + \frac{b}{d} =$ and $\frac{a}{c} - \frac{b}{d} =$
- $\frac{\frac{a}{b}}{\frac{c}{d}} =$


 **Warning.** $\frac{a}{c} + \frac{b}{d} = \frac{a+b}{c+d}$ is wrong!

We can use these rules to cancel fractions with a common factor on the top and bottom. For example,

$$\frac{2x+4y}{8} = \frac{2(x+2y)}{2 \cdot 4} = \frac{2}{2} \cdot \frac{x+2y}{4} = \frac{x+2y}{4}$$

or, more briefly,

$$\frac{2x+4y}{8} = \frac{\overset{2}{\cancel{2}x} + \underset{\cancel{4}}{\cancel{4}y}}{\underset{\cancel{8}}{\cancel{2} \cdot \cancel{4}}} = \frac{x+2y}{4}$$

 **Warning.** When trying to cancel, you have to make sure that you cancel common factor from every part of the fraction. For example, one potential mistake is to forget to cancel completely, like this:

$$\frac{2x+4y}{8} = \frac{\cancel{2}x + \cancel{4}y}{\underset{\cancel{8}}{\cancel{2} \cdot \cancel{4}}} = \frac{x+4y}{4}$$

is wrong.

Example 0.2.3. Compute $\frac{2}{\frac{3}{4}}$ and $\frac{\frac{2}{3}}{4}$.

Example 0.2.4. Express $\frac{\frac{2}{x}}{\frac{3}{xy}} + 7y$ as a single fraction.



Warning. If f is a function, then $f(x)$ means “the value of the function f at x ”. It does not mean “ f multiplied by x ”, because multiplying by a function has no meaning.

This means that you cannot apply rules for multiplication to expressions such as $f(x)$. For example, when x , a and b are variables, the distributive law tells us that $x(a+b) = xa + xb$. But when f is a function, it is (almost always) wrong to write $f(a+b) = f(a) + f(b)$ since

So it is very important to keep track of what is a function and what is not.

Example 0.2.5. If $f(x) = x + 1$, then $f(a+b) \neq f(a) + f(b)$. Why?

Example 0.2.6. We’ll meet two functions later which are called \sin and \cos . If s and c are variables, explain why one of these simplifications is correct, and the other is not:

$$\frac{s(\pi x)}{c(\pi x)} = \frac{s(\cancel{\pi x})}{c(\cancel{\pi x})} = \frac{s}{c} \quad \text{and} \quad \frac{\sin(\pi x)}{\cos(\pi x)} = \frac{\sin(\cancel{\pi x})}{\cos(\cancel{\pi x})} = \frac{\sin}{\cos}$$

0.3 Powers

If n is a positive integer, we write a^n for the product of a with itself n times:

$$a^n = \underbrace{a \times a \times \cdots \times a}_{n \text{ times}}$$

We also set $a^0 = 1$ and $a^{-n} = \frac{1}{a^n}$ if $a \neq 0$. For example, $a^1 = a$ and $a^{-1} = \frac{1}{a}$.

This means that we have defined a^n for any integer n . The expression a^n is read as “ a to the power of n ”. The number n is called the **exponent** and a is called the **base**.

We have the following **power laws for integer exponents**: if m and n are integers, then

- $(ab)^n = a^n \cdot b^n$ and $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$ and $\left(\frac{a}{b}\right)^{-n} = \frac{b^n}{a^n}$
- $a^m \cdot a^n = a^{m+n}$ and $\frac{a^m}{a^n} = a^{m-n}$
- $(a^m)^n = a^{mn}$

Example 0.3.1. Write $\left(\frac{2x^5}{3}\right)^{-3}$ without brackets.

One useful fact is that $(-1)^n = 1$ if n is an even integer like $-2, 0, 2, 4, 6, \dots$, and $(-1)^n = -1$ if n is an odd integer like $-3, -1, 1, 3, 5, 7, \dots$. This is helpful when dealing with powers and minus signs.

Example 0.3.2. Write the expressions $(-x)^n$ and $-(x^n)$ without brackets, where n is an integer.

Remark 0.3.3. Powers can be undone, under certain conditions, by operations known as **roots**. These include square roots and cube roots. We will discuss these in more detail later.

0.4 Order of operations

In an expression such as $1 + 2 \cdot 3$, there is one addition and one multiplication operation. However, there are no brackets, so it may be unclear whether this means that we should do the addition first, to get $(1 + 2) \cdot 3$, or that we should do the multiplication first, to get $1 + (2 \cdot 3)$.

To get around ambiguities like this, we adopt the following convention. Unless there are brackets which indicate otherwise, we perform operations in the following order:

1. powers and roots,
2. then negation, multiplication and division, working left to right,
3. then addition and subtraction, working left to right.

Some examples:

- $1 + 2x =$ and not $(1 + 2)x = 3x$, since we perform the multiplication of 2 and x before adding 1.
- $2x^3 =$ and not $(2x)^3 = (2x) \cdot (2x) \cdot (2x) = 8 \cdot x \cdot x \cdot x$, since cubing (taking power 3) comes before multiplication.

- $-2^2 =$ and not $(-2)^2 = 4$ since squaring (taking power 2) comes before negation.
- $3 - 2 - 1 =$ and not $3 - (2 - 1) = 3 - 1 = 2$, since we perform addition and subtraction working left to right.

The way powers are written means that special rules apply. For example, if we want to compute 3^{1-1} , since the $1 - 1$ is all in the exponent we should compute that first to get $3^{(1-1)} = 3^0 = 1$ (and not $(3^1) - 1 = 3 - 1 = 2$). Hopefully the fact that the exponent is written as a superscript will remind you that this has to be dealt with first. If you forget, it's a good idea to write the brackets in when you see an exponent involving more than one number or variable.



Warning. This special treatment of exponents means that when you write down an expression like 3^{1-1} , it must be clear that all of the exponent is a superscript. In other words, in this example, all of $1 - 1$ must be above the 3 so that it can't be mistaken for $3^1 - 1$ or $3 \cdot 1 - 1$.

Example 0.4.1. Simplify $x \cdot x^{k-1}$ where k is an integer.

Another special case is in stacked exponents like 2^{3^2} , which could be interpreted as $2^{(3^2)} = 2^9 = 512$ or $(2^3)^2 = 8^2 = 64$. In this case, we work from right to left, so the first interpretation is correct.

Example 0.4.2. Write $(-x^{y^2})^4$ without brackets.

0.5 Equations

An equation consists of two expressions separated by an “=” sign, such as

$$-2(x + 3) = 8.$$

A **solution** of an equation such as this is a real number such that when we substitute that number for x , the equation is satisfied: we get the same real numbers on each side of the equation. For example, $x = 2$ is not a solution of this equation, since

If an equation involves more than one variable, then a solution of the equation is a collection of real numbers, one for each variable, such that when we substitute the variables for these numbers, the equation is satisfied. An equation may have zero, one or several solutions.

Example 0.5.1. How many solutions does the equation $x = x + 1$ have?

Example 0.5.2. How many solutions does the equation $x^2 = 9$ have?

Two equations are **equivalent** if, no matter which numbers we substitute for the variables, either they are both satisfied, or they are both not satisfied. In other words, two equations are equivalent if they have exactly the same solutions. We indicate that two equations are equivalent using “ \iff ”.

Example 0.5.3. Explain why $-2(x + 3) = 8 \iff x + 3 = -4$.

If we apply the same operation (such as “divide by -2 ”) to both sides of an equation, and if the operation can be undone (for example, by “multiply by -2 ”) then we obtain a new equation which is equivalent to the first.

Example 0.5.4. Solve the equation $-2(x + 3) = 8$.
[This means: find all of the solutions].

The following operations can be undone:

- taking odd powers such as cubes, and the corresponding roots, such as cube roots,
- negation, multiplication and division, provided we’re sure that we don’t multiply or divide by 0,
- adding or subtracting.



Warning. If we take an equation and perform an operation that cannot be undone, then we usually won’t obtain an equivalent equation.

This means that we have to be careful when we multiply or divide by something that may be zero, or take even powers or roots (for example, squaring both sides of an equation).

Example 0.5.5. Explain what is wrong with this argument:

$$\begin{aligned}\sqrt{x} = -1 &\iff (\sqrt{x})^2 = (-1)^2 \text{ (square both sides)} \\ &\iff x = 1 \text{ (simplifying both sides).}\end{aligned}$$

So the solution to $\sqrt{x} = -1$ is $x = 1$.

Example 0.5.6. Explain what is wrong with this argument:

$$\begin{aligned}x(x+1) = 0 &\iff \frac{x(x+1)}{x+1} = 0 \text{ (divide both sides by } x+1) \\ &\iff x = 0 \text{ (by cancelling } x+1 \text{ on the left hand side).}\end{aligned}$$

So the solution to $x(x+1) = 0$ is $x = 0$.