Unexpected facets of the Yang-Baxter equation

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Yang-Baxter equation

✓ A vector space $V$ (or an object in any monoidal category)
✓ $\sigma: V \otimes^2 \rightarrow V \otimes^2$

Yang-Baxter equation (YBE):

$\sigma_1 \circ \sigma_2 \circ \sigma_1 = \sigma_2 \circ \sigma_1 \circ \sigma_2: V \otimes^3 \rightarrow V \otimes^3$

where $\sigma_i = \text{Id}_V^{\otimes i-1} \otimes \sigma \otimes \text{Id}_V^{\otimes \cdots}$.

A map $\sigma$ satisfying YBE is a braiding.
Yang-Baxter equation

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(REIDEMEISER III)
Yang-Baxter equation

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where $\sigma_i = \text{Id}^\otimes_{V} \circ \sigma \circ \text{Id}^\otimes_{V} \cdots$.

A map $\sigma$ satisfying YBE is a braiding.

History:

➡ Particle physics: factorization cond. for the dispersion matrix in the 1-dim. $n$-body problem (McGuire, Yang, 60’).

➡ Statistical mechanics: partition function for exactly solvable lattice models (Baxter, 70’).
Yang-Baxter equation

History:

→ **Particle physics**: factorization cond. for the dispersion matrix in the 1-dim. $n$-body problem (*McGuire, Yang, 60’*).

→ **Statistical mechanics**: partition function for exactly solvable lattice models (*Baxter, 70’*), quantum inverse scattering method for completely integrable systems (*Faddeev et al., 1979*).

→ **Field theories**: factorizable S-matrices in 2-dim. quantum field theory (*Zamolodchikov, 1979*), conformal field theory.

→ **Quantum groups** (*Drinfel’d, 80’*).

→ **C* algebras** (*Woronowicz, 80’*).

→ **Low-dimensional topology**.
A homology theory for the YBE

Aim: Unify homology theories for basic algebraic structures.
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Tools: Graphical calculus.
A homology theory for the YBE

**Aim**: Unify homology theories for basic algebraic structures.

**Tools**: Graphical calculus.

**Ingredients**:

- ✓ A braided vector space \((V, \sigma)\);
- ✓ a left **braided** \(V\)-module \((M, \rho: M \otimes V \to M)\) s.t.
  \[
  \rho \circ \rho_1 = \rho \circ \rho_1 \circ \sigma_2: M \otimes V \otimes V \to M
  \]
  \[\begin{array}{ccc}
  \rho & \rho & \rho \\
  M \otimes V \otimes V & = & M \otimes V \otimes V
  \end{array}\]
- ✓ a right braided \(V\)-module \((N, \lambda: V \otimes N \to N)\).
A homology theory for the YBE

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\[
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\]
\[
M \otimes V \otimes V \to M
\]

✓ a right braided \(V\)-module \((N, \lambda: V \otimes N \to N)\).

Theorem (L. 2013): \(M \otimes T(V) \otimes N\) carries a family of differentials \(\delta^{(\alpha, \beta)} = \alpha \cdot \delta + \beta \delta^*\), \(\alpha, \beta \in \mathbb{k}\).

(I.e., \(\delta^{(\alpha, \beta)} \circ \delta^{(\alpha, \beta)} = 0\)).
A homology theory for the YBE

**Theorem (L. 2013):** $M \otimes T(V) \otimes N$ carries a family of differentials $\delta^{(\alpha, \beta)} = \alpha \delta + \beta \delta^\bullet$, $\alpha, \beta \in k$.

\[ \delta = \sum (-1)^{i-1} \]
\[ \delta^\bullet = \sum (-1)^{i-1} \]
A homology theory for the YBE

**Theorem (L. 2013):** $M \otimes T(V) \otimes N$ carries a family of differentials $\delta^{(\alpha, \beta)} = \alpha \cdot \delta + \beta \cdot \delta^\circ$, $\alpha, \beta \in k$.

\[
\begin{align*}
\delta^\circ &= \sum (-1)^{i-1} \\
\delta &= \sum (-1)^{i-1}
\end{align*}
\]

**Proof:**

\[
\begin{align*}
\text{YBE} &\quad \equiv \quad \text{br. mod.} \\
\& \quad \text{sign} = (-1)^{\# \text{cross}}.
\end{align*}
\]
A homology theory for the YBE

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Remarks:
✓ Functoriality.
✓ Interpretation in terms of quantum shuffles (Rosso, 1995).
✓ Duality $\leadsto$ a cohomology theory.
✓ Pre-cubical structure.
A homology theory for the YBE

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✓ Degeneracies:

**Braided coalgebra:** br. v. sp. $(V, \sigma)$ & $\Delta: V \rightarrow V \otimes V$ s.t.

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
\includegraphics[width=1cm]{braided1}
\end{array}
\end{array}
\end{align*}
\]

(Cf. Reidemeister moves for knotted 3-valent graphs!)
A homology theory for the YBE

Theorem (L. 2013): $M \otimes T(V) \otimes N$ carries a family of differentials $\delta^{(\alpha, \beta)} = \alpha \cdot \delta + \beta \delta^\cdot$, $\alpha, \beta \in k$.

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$$\begin{align*}
\begin{array}{c}
\vspace{0.5cm}
\end{array}
\end{align*}$$

(Cf. Reidemeister moves for knotted 3-valent graphs!)

Theorem (L. 2013): All $\delta^{(\alpha, \beta)}$ restrict to $\sum_i \text{Im}(\Delta_i)$. $\Rightarrow$ normalization
Alg. structures via braidings

Associative algebras
A Associative algebras

\((V, \cdot, 1)\) s.t.

\[ \text{Associativity:} \quad (u \cdot v) \cdot w = u \cdot (v \cdot w) \]

\[ \text{Unit axiom:} \quad 1 \cdot v = v \cdot 1 = v \]
**Alg. structures via braidings**

**A** Associative algebras

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- **Associativity:**
  \[(u \cdot v) \cdot w = u \cdot (v \cdot w)\]

- **Unit axiom:**
  \[1 \cdot v = v \cdot 1 = v\]

\(\leadsto\) **"Associativity braiding"**

\[\sigma_{\text{Ass}} : v \otimes w \mapsto 1 \otimes v \cdot w\]

YBE for \(\sigma_{\text{Ass}} \iff \) associativity for \(\cdot\) (un. ax.)
3. Alg. structures via braidings

A. Associative algebras

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Proof:

$$\quad = \quad$$
3. Alg. structures via braidings

(A) Associative algebras

\[(V, \cdot, 1) \leadsto \sigma_{\text{Ass}}: v \otimes w \mapsto 1 \otimes v \cdot w\]

✓ YBE for \(\sigma_{\text{Ass}} \iff \text{associativity for } \cdot\)

✓ A fully faithful functor \(\text{Alg} \hookrightarrow \text{Br}.\)
### Alg. structures via braidings

#### A. Associative algebras

$$(V, \cdot, 1) \sim \sigma_{\text{Ass}} : v \otimes w \mapsto 1 \otimes v \cdot w$$

- YBE for $\sigma_{\text{Ass}} \iff$ (un. ax.) associativity for $\cdot$.
- A fully faithful functor $\text{Alg} \hookrightarrow \text{Br}$. 
- Duality: $\text{Alg} \leftrightarrow \text{Br.}$ \leftrightarrow \text{coAlg.}$
Assoc. algebras

\((V, \cdot, 1) \leadsto \sigma_{\text{Ass}}: v \otimes w \mapsto 1 \otimes v \cdot w\)

\(\checkmark \) YBE for \(\sigma_{\text{Ass}} \iff \) associativity for \(\cdot\) (un. ax.)

\(\checkmark \) A fully faithful functor \(\text{Alg} \hookrightarrow \text{Br.}\).

\(\checkmark \) Duality: \(\text{Alg} \leftrightarrow \text{Br.} \leftrightarrow \text{coAlg.}\).

\(\checkmark \) \(\sigma_{\text{Ass}} \circ \sigma_{\text{Ass}} = \sigma_{\text{Ass}} \Rightarrow \) highly non-invertible.

\(\checkmark \) Braided modules for \((V, \sigma_{\text{Ass}}) \leftrightarrow \) modules for \((V, \cdot, 1)\).
Alg. structures via braidings

A. Associative algebras

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✓ \(\sigma_{\text{Ass}} \circ \sigma_{\text{Ass}} = \sigma_{\text{Ass}} \implies \text{highly non-invertible.}\)

✓ Braided modules for \((V, \sigma_{\text{Ass}}) \leftrightarrow \text{modules for } (V, \cdot, 1)\).

✓ \(\Delta_{\text{Ass}}: v \mapsto 1 \otimes v \leadsto \text{braided coalgebra.}\)
Alg. structures via braidings

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✓ \(\Delta_{\text{Ass}} : v \mapsto 1 \otimes v \sim \text{braided coalgebra.} \)

✓ Braided homologies for \((V, \sigma_{\text{Ass}})\) include
  ➔ bar; ➔ Hochschild; ➔ group.
Leibniz algebras

\((V, [], 1)\) s.t.

- **Leibniz identity**: \([v, [w, u]] = [[v, w], u] - [[v, u], w]\);
- **Lie unit axiom**: \([1, v] = [v, 1] = 0\).

Alg. structures via braidings

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(Bloh 1965, Loday & Cuvier 1991: a non-commutative generalization of Lie algebras.)

\[ \sigma_{Lei} : v \otimes w \mapsto w \otimes v + 1 \otimes [v, w] \]

- **YBE for** \(\sigma_{Lei}\) \(\iff\) Leibniz identity for \([\cdot]\)

- A fully faithful functor \(\textbf{Lei} \hookrightarrow \textbf{Br}.\).
Alg. structures via braidings

Leibniz algebras

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\(\xrightarrow{\text{"Leibniz braiding"}}\)

\(\sigma_{\text{Lei}}: v \otimes w \mapsto w \otimes v + 1 \otimes [v, w]\)

- YBE for \(\sigma_{\text{Lei}} \iff\) Leibniz identity for \([\_\_\_]\)

- A fully faithful functor \(\text{Lei} \hookrightarrow \text{Br.}\).

- \(\sigma_{\text{Lei}}\) is invertible.

- Braided mod. for \((V, \sigma_{\text{Lei}}) \longleftrightarrow\) anti-symmetric Leibniz mod. for \((V, [], 1)\).
Leibniz algebras

\((V, [], 1)\) s.t. \([v, [w, u]] = [[v, w], u] - [[v, u], w], \quad [1, v] = [v, 1] = 0.\)

“Leibniz braiding”

\(\sigma_{\text{Lei}}: v \otimes w \mapsto w \otimes v + 1 \otimes [v, w]\)

Suppose that \(V = V' \oplus k1, [V', V'] \subseteq V'\). Then

\(\Delta_{\text{Lei}}: v \mapsto 1 \otimes v + v \otimes 1, v \in V'; \quad 1 \mapsto 1 \otimes 1\)

turns \((V, \sigma_{\text{Lei}})\) into a braided coalgebra.
Alg. structures via braidings

Leibniz algebras

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turns \((V, \sigma_{\text{Lei}})\) into a braided coalgebra.

✓ Braided homologies for \((V, \sigma_{\text{Lei}})\) include Leibniz homology.

\[
(M \otimes T(V), d_{\text{Lei}}) \quad \xrightarrow{\text{anti-symm.}} \quad (M \otimes \Lambda(V), d_{\text{CE}})
\]

Chevalley-Eilenberg

Cuvier-Loday
Leibniz algebras

\((V, [], 1)\) s.t. \([v, [w, u]] = [[v, w], u] - [[v, u], w], \quad [1, v] = [v, 1] = 0.\)

\(\leadsto \) “Leibniz braiding” \(\sigma_{\text{Lei}}: v \otimes w \mapsto w \otimes v + 1 \otimes [v, w]\)

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Braided homologies for \((V, \sigma_{\text{Lei}})\) include Leibniz homology.

\(\overset{\text{Lei}}{\text{Lie}}\) \(\overset{\text{anti-symm.}}{\overset{\text{V}}{\longrightarrow}}\) \(\overset{(M \otimes T(V), d_{\text{Lei}})}{\text{V}}\) \(\overset{\text{anti-symm.}}{\downarrow}\) \(\overset{(M \otimes \Lambda(V), d_{\text{CE}})}{\text{V}}\) \(\overset{\text{Cuvier-Loday}}{\downarrow}\) \(\overset{\text{Chevalley-Eilenberg}}{\downarrow}\)
Alg. structures via braidings

Leibniz algebras

\((V, [], 1)\) s.t. \([v, [w, u]] = [[v, w], u] - [[v, u], w], \quad [1, v] = [v, 1] = 0.\)

\(\leadsto \quad \text{“Leibniz braiding”} \quad \sigma_{\text{Lei}}: v \otimes w \mapsto w \otimes v + 1 \otimes [v, w]\)

✓ Suppose that \(V = V' \oplus k1, [V', V'] \subseteq V'.\) Then

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✓ Braided homologies for \((V, \sigma_{\text{Lei}})\) include Leibniz homology.

\begin{align*}
\text{Lei} & \quad V \rightarrow (M \otimes T(V), d_{\text{Lei}}) \\
\text{Lie} & \quad V \rightarrow (M \otimes \Lambda(V), d_{\text{CE}})
\end{align*}

✓ Explains the choice of the lift of the Jacobi identity.
Alg. structures via braidings

Self-distributive structures
Alg. structures via braidings

Self-distributive structures

\[ b \triangleleft a \triangleleft b \]

\[ \text{colorings by } (S, \triangleleft) \]

\[ a \triangleleft b \]

\[ a \triangleleft b \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c) \]

(SD)
C. Self-distributive structures

\[ \sigma_{SD}: (a, b) \mapsto (b, a \triangleleft b) \]

"SD braiding"

YBE $\longleftrightarrow$ RIII $\longleftrightarrow$ \((a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)\) (SD)
Alg. structures via braidings

Self-distributive structures

\[ b \prec a \prec b \]

colorings by \((\mathcal{S}, \prec)\)

"SD braiding"

\[ \sigma_{SD}: (a, b) \mapsto (b, a \prec b) \]

\[ \sigma^{-1}_{SD} \]

YBE \[ \iff \] RIII \[ \iff \] (a \prec b) \prec c = (a \prec c) \prec (b \prec c) \quad \text{(SD)}

\[ \exists \sigma_{SD} \]

RII \[ \iff \] a \mapsto a \prec b \text{ bijective} \quad \text{(Inv)}

RI \[ \iff \] a \prec a = a \quad \text{(Idem)}
Alg. structures via braidings

C Self-distributive structures

![Diagram showing self-distributive structures and braidings]

\[ \Delta_{SD}: a \mapsto (a, a) \]

\[ \exists \sigma^{-1}_{SD} \longleftrightarrow \text{RII} \longleftrightarrow a \mapsto a \triangleleft b \text{ bijective} \]

\[ \exists \gamma \equiv \gamma \longleftrightarrow \text{RI} \longleftrightarrow a \triangleleft a = a \]

YBE \longleftrightarrow RIII \longleftrightarrow (a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c) \quad \text{(SD)}

\[ (\text{Inv}) \]

\[ (\text{Idem}) \]
Alg. structures via braidings

Self-distributive structures

Joyce, Matveev 1982:

knot invariants $\leadsto$ quandle

| pos. braids | RIII | $\leftrightarrow$ | $(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$ | shelf |
| braids | RII | $\leftrightarrow$ | $a \mapsto a \triangleleft b$ bijective | rack |
| knots | RI | $\leftrightarrow$ | $a \triangleleft a = a$ | quandle |

\[
\begin{array}{c}
\begin{array}{c}
\uparrow \quad \uparrow \\
\downarrow \quad \downarrow \\
R_{III} & \leftrightarrow & R_{II} & \leftrightarrow & R_{I}
\end{array}
\end{array}
\]
### Alg. structures via braidings

#### C Self-distributive structures

- **Colorings** by \((S, \triangleleft)\)

Joyce, Matveev 1982: knot invariants \(\leadsto\) quandle

<table>
<thead>
<tr>
<th>pos. braids</th>
<th>RIII</th>
<th>( (a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c) )</th>
<th>shelf</th>
</tr>
</thead>
<tbody>
<tr>
<td>braids</td>
<td>RII</td>
<td>( a \mapsto a \triangleleft b ) bijective</td>
<td>rack</td>
</tr>
<tr>
<td>knots</td>
<td>RI</td>
<td>( a \triangleleft a = a )</td>
<td>quandle</td>
</tr>
</tbody>
</table>

**Ex.:** **Conjugation quandles:** (group \(G, g \triangleleft h = h^{-1}gh\))

| Coloring rule | Wirtinger presentation rule, colorings | \( \leadsto \) \( \text{Rep}(\pi_1(\mathbb{R}^3 \setminus K), G) \). |

*Colorings* \(\leadsto\) *quandle*
Alg. structures via braidings

Self-distributive structures

Joyce, Matveev 1982:

knot invariants $\leadsto$ quandle

colorings $\leadsto$ quandle

coloring rule $\leadsto$ Wirtinger presentation rule,

colorings $\leadsto$ Rep$(\pi_1(\mathbb{R}^3 \setminus K), G)$.

Ex.: Conjugation quandles: (group $G$, $g \triangleleft h = h^{-1}gh$)

Dihedral quandles: ($\mathbb{Z}_n$, $a \triangleleft b = 2b - a$)

colorings $\leadsto$ $n$-Fox colorings.
Alg. structures via braidings

Self-distributive structures

Joyce, Matveev 1982:

Knot invariants $\leadsto$ quandle

colorings $\leadsto$ quandle

- **pos. braids** $\longleftrightarrow$ RIII $\leadsto$ $(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$
- **braids** $\longleftrightarrow$ RII $\leadsto$ $a \mapsto a \triangleleft b$ bijective
- **knots** $\longleftrightarrow$ RI $\leadsto$ $a \triangleleft a = a$

**Ex.**

- **Conjugation quandles**: $(\text{group } G, g \triangleleft h = h^{-1} gh)$
  - coloring rule $\leadsto$ Wirtinger presentation rule,
  - colorings $\leadsto$ $\text{Rep}(\pi_1(\mathbb{R}^3 \setminus K), G)$.

- **Dihedral quandles**: $(\mathbb{Z}_n, a \triangleleft b = 2b - a)$
  - colorings $\leadsto$ $n$-Fox colorings.

$n = 3$ $\neq$ $\bigcirc$
Self-distributive structures

diagrams: \( D \xrightarrow{\text{R-move}} D' \)
colorings: \( \mathcal{C} \xrightarrow{\sim} \mathcal{C}' \)
coloring sets: \( \text{Col}_S(D) \xleftrightarrow{\text{bij.}} \text{Col}_S(D') \)
counting invariants: \( \#\text{Col}_S(D) = \#\text{Col}_S(D') \)
Alg. structures via braidings

Self-distributive structures

- diagrams: \( D \xrightarrow{\text{R-move}} D' \)
- colorings: \( C \xrightarrow{\sim} C' \)
- coloring sets: \( \text{Col}_S(D) \leftrightarrow \text{bij.} \text{Col}_S(D') \)
- counting invariants: \( \# \text{Col}_S(D) = \# \text{Col}_S(D') \)

Question: Extract more information?

Idea: Some "weight" \( \omega \) s.t. \( \omega(C) = \omega(C') \)

\[ \Rightarrow \{ \omega(C) | C \in \text{Col}_S(D) \} = \{ \omega(C') | C' \in \text{Col}_S(D') \}. \]
Alg. structures via braidings

Self-distributive structures

- **Diagrams**: $D \xrightarrow{\text{R-move}} D'$
- **Colorings**: $C \xrightarrow{\text{bij.}} C'$
- **Coloring sets**: $\text{Col}_S(D) \leftrightarrow \text{Col}_S(D')$
- **Counting invariants**: $\# \text{Col}_S(D) = \# \text{Col}_S(D')$

**Question**: Extract more information?

**Idea**: Some “weight” $\omega$ s.t. $\omega(C) = \omega(C')$ implies $\{\omega(C)| C \in \text{Col}_S(D)\} = \{\omega(C')| C' \in \text{Col}_S(D')\}$.

**Answer**: **Quandle cocycle invariants** (Carter-Jelsovsky-Kamada-Langford-Saito 1999).

- $\phi : S \times S \to A \xrightarrow{\sim}$
- **Boltzmann weight**: $\omega_\phi(C) = \sum_{a,b} \pm \phi(a, b)$
Alg. structures via braidings

Self-distributive structures

Rack & quandle cohomology theories

(Fenn-Rourke-Sanderson 1995, Carter et al. 1999)

Motivation:

\[ \{ \omega_\phi(C) | C \in \text{Col}_S(D) \} \] yields a braid / knot invariant when \( \phi \) is a rack / quandle 2-cocycle;

\( \Rightarrow \) this invariant is trivial when \( \phi \) is a coboundary;
Alg. structures via braidings

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\( \Rightarrow \) this invariant is trivial when \( \phi \) is a coboundary;

\( \Rightarrow \) adding coefficients allows to color diagram regions:

\[
\omega_\phi(C) = \sum_{a, m, b} \pm \phi(m, a, b)
\]

\( \Rightarrow \) everything generalizes to \( K^{n-1} \hookrightarrow \mathbb{R}^{n+1} \).
Alg. structures via braidings

C) Self-distributive structures

Rack & quandle cohomology theories

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adding coefficients allows to color diagram regions:

\[ \omega_\phi(\mathcal{C}) = \sum_{a,b,m} \pm \phi(m,a,b) \]

everything generalizes to \( K^{n-1} \hookrightarrow \mathbb{R}^{n+1} \).

Inspiration: (Co-)homology theories for associativity.

Question (Przytycki): Explain the parallels between the associative and the SD worlds?
Alg. structures via braidings

Self-distributive structures

Rack & quandle cohomology theories

(Fenn-Rourke-Sanderson 1995, Carter et al. 1999)

Motivation:

- \( \{ \omega_\phi(C) | C \in \text{Col}_S(D) \} \) yields a braid / knot invariant when \( \phi \) is a rack / quandle 2-cocycle;
- this invariant is trivial when \( \phi \) is a coboundary;
- adding coefficients allows to color diagram regions:

\[
\omega_\phi(C) = \sum_{a, m, b} \pm \phi(m, a, b)
\]

- everything generalizes to \( K^{n-1} \hookrightarrow \mathbb{R}^{n+1} \).

Inspiration: (Co-)homology theories for associativity.

Question (Przytycki): Explain the parallels between the associative and the SD worlds?
Answer: Common braided interpretation.
Alg. structures via braidings

Self-distributive structures

\[ \text{Shelf } (S, \triangleleft) \leadsto \sigma_{SD}: (a, b) \mapsto (b, a \triangleleft b) \]

✓ YBE for \( \sigma_{SD} \iff \text{SD for } \triangleleft \)

✓ A fully faithful functor \( \text{Shelf} \hookrightarrow \text{Br} \).

✓ \( \sigma_{SD} \) is invertible \( \iff (S, \triangleleft) \) is a rack.

✓ Braided modules for \( (V, \sigma_{SD}) \) \( \iff \) rack modules for \( (S, \triangleleft) \).

✓ \( \Delta_{SD}: a \mapsto (a, a) \) \( \leadsto \) weak braided coalgebra if \( (S, \triangleleft) \) is a quandle.

✓ Braided homologies for \( (V, \sigma_{SD}) \) include rack, quandle, and other SD homologies.
Multi-component braidings

Question: How to treat more complicated structures?
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Braided system: $V_1, V_2, \ldots, V_r$ and $\sigma^{i,j}: V_i \otimes V_j \rightarrow V_j \otimes V_i$, $i \leq j$, satisfying the colored Yang-Baxter equation (cYBE):

$$\sigma^{j,k}_1 \circ \sigma^{i,k}_2 \circ \sigma^{i,j}_1 = \sigma^{i,j}_2 \circ \sigma^{i,k}_1 \circ \sigma^{j,k}_2$$

$$V_i \otimes V_j \otimes V_k \rightarrow V_k \otimes V_j \otimes V_i, \ i \leq j \leq k$$

The collection $(\sigma^{i,j})$ satisfying cYBE is a multi-braiding.
Multi-component braidings

Question: How to treat more complicated structures?

Braided system: $V_1, V_2, \ldots, V_r$ and $\sigma^{i,j}: V_i \otimes V_j \to V_j \otimes V_i$, $i \leq j$, satisfying the colored Yang-Baxter equation (cYBE):

$$\sigma_{1}^{j,k} \circ \sigma_{2}^{i,k} \circ \sigma_{1}^{i,j} = \sigma_{2}^{i,j} \circ \sigma_{1}^{i,k} \circ \sigma_{2}^{j,k}$$

$$V_i \otimes V_j \otimes V_k \to V_k \otimes V_j \otimes V_i, i \leq j \leq k$$

The collection $(\sigma^{i,j})$ satisfying cYBE is a multi-braiding.

Left braided $V$-module: $(M, (\rho_i: M \otimes V_i \to M))$ s.t.

$$\rho_j \circ \rho_i = \rho_i \circ \sigma_{i,j}$$

$i \leq j$
Multi-component braidings

Question: How to treat more complicated structures?

Braided system: \(V_1, V_2, \ldots, V_r\) and \(\sigma^{i,j}: V_i \otimes V_j \rightarrow V_j \otimes V_i, \ i \leq j\), satisfying the **colored Yang-Baxter equation (cYBE):**

\[
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\]

\(V_i \otimes V_j \otimes V_k \rightarrow V_k \otimes V_j \otimes V_i, \ i \leq j \leq k\)

The collection \((\sigma^{i,j})\) satisfying cYBE is a **multi-braiding**.

Left braided \(V\)-module: 

\((M, (\rho_i: M \otimes V_i \rightarrow M))\) s.t.

\[
\rho_j \rho_i = \rho_i \rho_j \sigma^{i,j}
\]

\(i \leq j\)

**Theorem (L., 2013):** \(M \otimes T(V_1) \otimes \cdots \otimes T(V_r) \otimes N\) carries a family of differentials \(\delta^{(\alpha, \beta)} = \alpha \cdot \delta + \beta \cdot \delta^*\), \(\alpha, \beta \in \mathbb{k}\).
Multi-component braidings

Finite-dim. **bialgebra** $H \leadsto (H, H^*; \sigma_{H,H} = \sigma_{\text{Ass}}^r(H), \sigma_{H^*,H^*} = \sigma_{\text{Ass}}(H^*), \sigma_{H,H^*} = \sigma_{\text{YD}})$

\[
\sigma_{H,H} = \begin{array}{c}
\cdot \\
\cdot
\end{array} \\
\sigma_{H^*,H^*} = \begin{array}{c}
\cdot \\
\cdot
\end{array} \\
\sigma_{H,H^*} = \begin{array}{c}
\cdot \\
\cdot
\end{array}
\]

\[
h \otimes l \mapsto \langle l_{(1)}, h_{(2)} \rangle l_{(2)} \otimes h_{(1)}
\]

✓ **YBE on** $H \otimes H^* \otimes H^*$ \iff **bialgebra compatibility** (un. ax.)
Multi-component braidings

Finite-dim. bialgebra $H \sim (H, H^*; \sigma_{H,H} = \sigma^r_{\text{Ass}}(H), \sigma_{H^*,H^*} = \sigma_{\text{Ass}}(H^*), \sigma_{H,H^*} = \sigma_{\text{YD}})$

\[ \sigma_{H,H} = \begin{array}{c} \cdot \\ \end{array} \quad \sigma_{H^*,H^*} = \begin{array}{c} \cdot \\ \end{array} \quad \sigma_{H,H^*} = \begin{array}{c} \cdot \\ \end{array} \]

\[ h \otimes l \mapsto \langle l_{(1)}, h_{(2)} \rangle l_{(2)} \otimes h_{(1)} \]

✓ YBE on $H \otimes H^* \otimes H^*$ $\iff$ bialgebra compatibility (un. ax.)

✓ A fully faithful functor $\mathfrak{Bialg} \hookrightarrow \mathfrak{2BrSyst}$.
Multi-component braidings

Finite-dim. bialgebra $H \leadsto (H, H^*; \sigma_{H, H} = \sigma^r_{\text{Ass}}(H), \sigma_{H^*, H^*} = \sigma_{\text{Ass}}(H^*), \sigma_{H, H^*} = \sigma_{\text{YD}})$

$\sigma_{H, H} = \begin{array}{c} \bullet \\ \downarrow \end{array}$  $\sigma_{H^*, H^*} = \begin{array}{c} \bullet \\ \downarrow \end{array}$  $\sigma_{H, H^*} = \begin{array}{c} \bullet \\ \downarrow \end{array}$

$h \otimes l \mapsto \langle l_{(1)}, h_{(2)} \rangle l_{(2)} \otimes h_{(1)}$

✓ YBE on $H \otimes H^* \otimes H^*$ $\iff$ bialgebra compatibility

(un. ax.)

✓ A fully faithful functor $^{*}\text{Bialg} \hookrightarrow ^{*}\text{BrSyst}$.

✓ $\sigma_{H, H^*}$ is invertible $\iff$ $H$ is a Hopf algebra.
Multi-component braidings

Finite-dim. bialgebra $H \leadsto (H, H^*; \sigma_{H,H} = \sigma_{\text{Ass}}^r(H), \sigma_{H^*,H^*} = \sigma_{\text{Ass}}(H^*), \sigma_{H,H^*} = \sigma_{\text{YD}})$

\[
\sigma_{H,H} = \begin{pmatrix} \cdot \end{pmatrix} \quad \sigma_{H^*,H^*} = \begin{pmatrix} \cdot \end{pmatrix} \quad \sigma_{H,H^*} = \begin{pmatrix} \cdot \end{pmatrix}
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✓ Braided modules $\leftrightarrow$ Hopf modules over $H$. 
Multi-component braidings

Finite-dim. bialgebra $H \leadsto (H, H^*; \sigma_{H,H} = \sigma_{\text{Ass}}(H), \sigma_{H^*,H^*} = \sigma_{\text{Ass}}(H^*), \sigma_{H,H^*} = \sigma_{YD})$

$\sigma_{H,H} = \begin{tikzpicture} [scale=0.5]
    
ode at (0,0) {$\otimes$};
    
ode at (2,0) {$\otimes$};
    
ode at (0,-2) {$\otimes$};
    
ode at (2,-2) {$\otimes$};
    
ode at (1,-1) {$\cdot$};
    
ode at (1,-3) {$\cdot$};

\end{tikzpicture}$

$\sigma_{H^*,H^*} = \begin{tikzpicture} [scale=0.5]
    
ode at (0,0) {$\otimes$};
    
ode at (2,0) {$\otimes$};
    
ode at (0,-2) {$\otimes$};
    
ode at (2,-2) {$\otimes$};
    
ode at (1,-1) {$\cdot$};
    
ode at (1,-3) {$\cdot$};

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✓ YBE on $H \otimes H^* \otimes H^*$ $\iff$ bialgebra compatibility (un. ax.)

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✓ $\sigma_{H,H^*}$ is invertible $\iff$ $H$ is a Hopf algebra.

✓ Braided modules $\iff$ Hopf modules over $H$.

✓ Braided homologies include

$\rightarrow$ Gerstenhaber-Schack;

$\rightarrow$ Panaite-$Ş$tefan.
Multi-component braidings

Finite-dim. bialgebra $H \sim (H, H^{op}, H^*, (H^*)^{op}; ...)$.

✓ A fully faithful functor $\ast \text{Bialg} \hookrightarrow \ast \mathcal{BrSyst}$. 
Multi-component braidings

Finite-dim. bialgebra $H \sim (H, H^{\text{op}}, H^*, (H^*)^{\text{op}}; \ldots)$.

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Multi-component braidings

Finite-dim. bialgebra $H \rightsquigarrow (H, H^{\text{op}}, \text{H}^*, (\text{H}^*)^{\text{op}}; \ldots)$.

✓ A fully faithful functor $\ast \text{Bialg} \hookrightarrow \ast \text{BrSyst}^\bullet$.

✓ Braided modules $\longleftrightarrow$ Hopf bimodules over $H$.

**Application:**

$\Rightarrow$ Hopf bimodules are modules over the Heisenberg double

$$\mathcal{H}(H) = H \otimes \text{H}^*$$

$\Rightarrow$ Cibils-Rosso 1998: “Hopf bimodules are modules” over

$$\mathcal{K}(H) = (H \otimes H^{\text{op}}) \otimes (\text{H}^* \otimes (\text{H}^*)^{\text{op}})$$
Multi-component braidings

Finite-dim. bialgebra $H \leadsto (H, H^{\text{op}}, H^*, (H^*)^{\text{op}}; \ldots)$.

✓ A fully faithful functor $^*\text{Bialg} \xhookrightarrow{\_} ^*\text{BrSyst}$. 

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Application:

→ Hopf bimodules are modules over the Heisenberg double

$\mathcal{H}(H) = H \otimes H^*$

→ Cibils-Rosso 1998: “Hopf bimodules are modules” over

$\mathcal{K}(H) = (H \otimes H^{\text{op}}) \otimes (H^* \otimes (H^*)^{\text{op}})$

→ Panaite 2002: “Hopf bimodules are modules over …”

$\mathcal{U}(H) = H^* \# (H^{\text{op}} \otimes H) \# (H^*)^{\text{op}}$ \&

$\mathcal{L}(H) = (H^* \otimes (H^*)^{\text{op}}) \Join (H^{\text{op}} \otimes H)$
Multi-component braidings

Finite-dim. bialgebra $H \sim (H, H^{op}, H^{*}, (H^{*})^{op}; ...)$.

✓ A fully faithful functor $\ast \text{Bialg} \hookrightarrow \ast \text{BrSyst}$.

✓ Braided modules $\leftrightarrow$ Hopf bimodules over $H$.

Application:

$\rightarrow$ Hopf bimodules are modules over the Heisenberg double $\mathcal{H}(H) = H \otimes H^{*}$

$\rightarrow$ Cibils-Rosso 1998: “Hopf bimodules are modules” over $\mathcal{K}(H) = (H \otimes H^{op}) \otimes (H^{*} \otimes (H^{*})^{op})$

$\rightarrow$ Panaite 2002: “Hopf bimodules are modules over ...” $\mathcal{Y}(H) = H^{*} \# (H^{op} \otimes H) \# (H^{*})^{op}$ & $\mathcal{Z}(H) = (H^{*} \otimes (H^{*})^{op}) \ltimes (H^{op} \otimes H)$

$\rightarrow$ Theorem (L. 2013): Hopf bimodules are modules over $4! = 24$ pairwise isomorphic algebras.
Multi-component braidings

Finite-dim. bialgebra \( \mathbf{H} \sim (\mathbf{H}, \mathbf{H}^\text{op}, \mathbf{H}^*, (\mathbf{H}^*)^\text{op}; \ldots) \).

✓ A fully faithful functor \( \ast \text{Bialg} \hookrightarrow \ast \text{BrSyst}^\bullet \).

✓ Braided modules \( \leftrightarrow \) Hopf bimodules over \( \mathbf{H} \).

Application:

⇒ Hopf bimodules are modules over the Heisenberg double
\[
\mathcal{H}(\mathbf{H}) = \mathbf{H} \otimes \mathbf{H}^*
\]

⇒ Cibils-Rosso 1998: “Hopf bimodules are modules” over
\[
\mathcal{K}(\mathbf{H}) = (\mathbf{H} \otimes \mathbf{H}^\text{op}) \otimes (\mathbf{H}^* \otimes (\mathbf{H}^*)^\text{op})
\]

⇒ Panaite 2002: “Hopf bimodules are modules over ...”
\[
\mathcal{V}(\mathbf{H}) = \mathbf{H}^* \# (\mathbf{H}^\text{op} \otimes \mathbf{H}) \# (\mathbf{H}^*)^\text{op}
\quad \&
\mathcal{L}(\mathbf{H}) = (\mathbf{H}^* \otimes (\mathbf{H}^*)^\text{op}) \Join (\mathbf{H}^\text{op} \otimes \mathbf{H})
\]

⇒ Theorem (L. 2013): Hopf bimodules are modules over \( 4! = 24 \) pairwise isomorphic algebras.

✓ Braided homologies include the Ospel-Taillefer theory.
The unifying role of the YBE

<table>
<thead>
<tr>
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### The unifying role of the YBE

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Other “braidable” structures:
- module algebras (Yau);
- Yetter-Drinfel’d modules (Panaite-Ştefan);
The unifying role of the YBE

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Other “braidable” structures:
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- (non-commutative) Poisson algebras (Fresse);
The unifying role of the YBE

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Other "braidable" structures:

- module algebras (Yau);
- Yetter-Drinfel’d modules (Panaite-Ştefan);
- (non-commutative) Poisson algebras (Fresse);
- multiple conjugation quandles (Ishii)

knotted handle-bodies
A braided version of YD modules

Yetter-Drinfel’d module over a Hopf algebra $H$:

✓ vector space $M$;
✓ right $H$-action $\rho$;
✓ right $H$-coaction $\delta$;
✓ compatibility.
A braided version of YD modules

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\[ \leadsto \text{braiding } \sigma_{\text{YD}}: \]
A braided version of YD modules

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This construction yields all invertible f.-d. braidings!
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L.-Wagemann 2015: YD module over a br. system $(C, A; \sigma)$:

- vector space $M$;
- right action $\rho$ of $(A; \sigma_A, A)$;
- right coaction $\delta$ of $(C; \sigma_C, C)$;
- compatibility:

\[
\begin{align*}
\delta & \quad \rho \\
M & \quad A & M & \quad C
\end{align*}
\]

\[
\begin{align*}
\delta & \quad \sigma_{C,A} \\
M & \quad A & M & \quad C
\end{align*}
\]
A braided version of YD modules

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**L.-Wagemann 2015**: **YD module over a br. system** $(C,A;\sigma)$:

- vector space $M$;
- right action $\rho$ of $(A;\sigma_A,A)$;
- right coaction $\delta$ of $(C;\sigma_C,C)$;
- compatibility:

$$\delta \circ \rho = \delta \circ \rho \circ \sigma_{C,A}$$

& a “nice” $\pi: C \rightarrow A$

$$\leadsto$$ braiding $\sigma_{GYD}$:
A braided version of YD modules

YD module over a braided system \((C, A; \sigma)\):

- ✓ vector space \(M\);
- ✓ right action \(\rho\) of \((A; \sigma_{A,A})\);
- ✓ right coaction \(\delta\) of \((C; \sigma_{C,C})\);
- ✓ compatibility:

\[
\delta \rho_C M A = \delta \rho_C M A \sigma_{C,A} \]

& a “nice” \(\pi: C \to A\)

\(\leadsto\) braiding \(\sigma_{GYD}\):

Examples:

\(\leadsto\) YD modules over a Hopf algebra \(\sim \sigma_{YD}\)
A braided version of YD modules

YD module over a braided system $(C, A; \sigma)$:

- vector space $M$;
- right action $\rho$ of $(A; \sigma_{A,A})$;
- right coaction $\delta$ of $(C; \sigma_{C,C})$;
- compatibility:

$$\delta \rho C A M M = \delta \rho C A M M \sigma_{C,A}$$

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$\leadsto$ braiding $\sigma_{GYD}$:

Examples:

- YD modules over a Hopf algebra $\leadsto \sigma_{YD}$;
- representations of a crossed module of groups $\leadsto$

Bantay’s braiding (2010);
A braided version of YD modules

YD module over a braided system \((\mathcal{C}, \mathcal{A}; \sigma)\):

- ✓ vector space \(M\);
- ✓ right action \(\rho\) of \((\mathcal{A}; \sigma_{\mathcal{A}, \mathcal{A}})\);
- ✓ right coaction \(\delta\) of \((\mathcal{C}; \sigma_{\mathcal{C}, \mathcal{C}})\);
- ✓ compatibility:


\[
\begin{align*}
\delta \rho & = \delta \rho \\
\delta & = \delta
\end{align*}
\]

& a “nice” \(\pi: \mathcal{C} \rightarrow \mathcal{A}\)

\sim braiding \(\sigma_{GYD}\):

Examples:

- ➤ YD modules over a Hopf algebra \(\sim \sigma_{YD}\);
- ➤ representations of a crossed module of groups \(\sim\)

Bantay’s braiding (2010);
- ➤ shelf \(\sim \sigma_{SD}\);
A braided version of YD modules

YD module over a braided system \((C, A; \sigma)\):

✓ vector space \(M\);
✓ right action \(\rho\) of \((A; \sigma_A, A)\);
✓ right coaction \(\delta\) of \((C; \sigma_C, C)\);
✓ compatibility:

\[
\delta \rho_{C, A} M = \delta \rho_{C, A} M \sigma_{C, A}
\]

& a “nice” \(\pi: C \to A\)

\(\sim\) braiding \(\sigma_{GYD}\):

Examples:

→ YD modules over a Hopf algebra \(\sim \sigma_{YD}\);
→ representations of a crossed module of groups \(\sim\)

Bantay's braiding (2010);
→ shelf \(\sim \sigma_{SD}\);
→ representations of a crossed module of shelves / Leibniz algebras \(\sim\) new braidings (L.-Wagemann 2015).