8) Motivations & definitions

Gavin Wraith's school-time riddles:

1. \[(P_2)R = \frac{PR}{(R_2)R}\]

2. \[\alpha, \beta, \gamma \in S_5\]
   \[\alpha \beta = \beta^{-1} \alpha \beta\]

Definition: \((A, *)\) is called:

- **shelf** if \(R_3\)
- **(w)rack** if \(R_3 \& R_2\)
- **quandle** if \(R_3 \& R_2 \& R_1\)

- \(R_2\) \(\forall \alpha \in A, \) the right translation \(\alpha \rightarrow \alpha \ast \beta\) is a bijection \(A \rightarrow A\)
- \(R_3\) \(\forall \alpha \in A, \) \(\alpha \ast \alpha = \alpha\)
Examples

1. Alexander quandle: $A \in \mathbb{Z}/j \mathbb{Z}$ Mod, $a \ast b = a + (1 - t)b$.

2. Conjugation quandle: group $G$, $a \ast b = b^{-1}ab$.
   Remark: free quandles come from these.

3. Core quandle: group $G$, $a \ast b = ba^{-1}b$.

4. Coxeter rack: $V = k^n$ a "good" form \sim $A = V \setminus \{0\}$, $a \ast b = a - 2 \frac{(a, b)}{(b, b)} b$.

5. Free shelf on $\alpha$ \sim a total order on braid groups (Dehornoy '94).

6. Free shelf on $\alpha$

   \[ \begin{array}{c}
   a \ast (a \ast (a \ast a)) = a \\
   2^n + 4 a's \end{array} \]
   is a finite shelf, related to large cardinals.

   Layer's n-table
Knot invariants

Reidemeister \( \text{R}_1 \): Knots = Diagrams / \( \text{R}_1 \) - \( \text{R}_3 \)

\[ \text{R}_1: \begin{array}{c} \gamma = 1 \equiv \gamma \\ \end{array} \]

\[ \text{R}_2: \begin{array}{c} \gamma = 11 \equiv \gamma \\ \end{array} \]

\[ \text{R}_3: \begin{array}{c} \gamma = \gamma \\ \end{array} \]

\( \text{Ex: trefoil knot} \)

\( T = b_0 \)

\( Q_T = \langle a, b, c \rangle / a = c a b, b = a a a c, c = b a a \)

\( \Rightarrow \langle a, b, c \rangle / a = c a b, b = (a a b)^4 a \)

Knot \( K \Rightarrow \) diagram \( D_K \Rightarrow \) quandle \( Q_K \) (knot quandle)

- generators = arcs
- relations: \( a \Rightarrow a b \)

Prop: \( Q_K \) does not depend on the choice of the diagram \( D_K \).

\[ \text{Dia. 3:} \begin{array}{c} b a c \equiv (a a b) a c \\ a b c \\ b a c \equiv (a a c) a (b a c) \\ a b c \\ a b c \\ \end{array} \]

Thm (Joyce, Matveev '82): \( Q_K \) is a weak universal knot invariant.

\( Q_K \cong Q_K' \Rightarrow K = K' \) or \( K, K' \) mirror.

Rem.: Universal enveloping group of the knot group.

P6: Quandles are difficult to compare.

Solution: Consider \( A \)-colourings of \( K \), i.e. \( \text{Rep} (Q_K, A) \).

\( n_K, A \in \mathbb{Z} \) more information? see next part!

\( \text{Ex.:} A = \mathbb{Z}_3, a \cdot b = 2 b - a \)

\( a \rightarrow a b \) either \( a = b = a a b \), or they are all different.

\( \Rightarrow \) Fox colourings

\( n_{\text{red}} = 3 \Rightarrow \text{trefoil knot} \)

\( n_{\text{red}} = 3 + 3 = 6 \Rightarrow 4 \) unknot
Rack cohomology of a rack $A$ is the cohomology of the complex

$$C^k(A, \mathbb{Z}) = \text{Map}(A^k, \mathbb{Z})$$

or any abelian group or twisted coefficients.

$$d^n: (a_1, ..., a_{n+1}) \mapsto \sum_{i=1}^{n+1} (-1)^i [f(a_1, a_2, ..., a_i, a_{i+1}, ..., a_{n+1}) - f(a_1, a_2, ..., a_{i-1}, a_i, a_{i+1}, ..., a_{n+1})]$$

If $A$ is a quandle, then $C^\infty(A, \mathbb{Z}) = \{ f: A^\infty \to \mathbb{Z} | f(\ldots, a, a, \ldots) = 0 \}$ is a subcomplex. Its cohomology is called the \textit{quandle cohomology} of $A$.

Applications:

1. For $w \in H^2_0(A, \mathbb{Z})$, the multiset $\{ BW_w(C) | C \in \text{Rep}(R_k, A) \}$ is a knot invariant.

Boltzmann weights:

\[
\begin{array}{c c c c c}
\otimes & 0 & 0 & 0 & 0 \\
\otimes & 0 & 0 & 0 & 0 \\
\end{array}
\]

This invariant can distinguish $K$ from $-K$.

2. Computation of $H^*_k(A, \mathbb{Z})$ for certain racks is an important step in classifying pointed Hopf algebras.
Yang-Baxter equation

\[ \sigma: A \times A \to A \times A \]

Rack \( A \) \( \overset{\sigma}{\to} \) \( A \times A \) \( \sigma: a \times b \mapsto (a \sigma b, a \sigma b) \)

\[ \sigma = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \]

\[ \sigma = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \]

Solution to the YBE \( \sigma_1 \sigma_2 \sigma_3 = \sigma_3 \sigma_2 \sigma_1 \)

\[ \sigma_1 = \sigma \times \text{Id}_A, \sigma_2 = \text{Id}_A \times \sigma \]

R3 move.

Other sources of solutions to the YBE:

- Monoid \( \sigma(0, b) = (1, 0, b) \)
- YBE \( \Rightarrow \) associativity

- Lie algebra \( \sigma(0 \theta) = \theta [0, b] + b \theta \)
- YBE \( \Rightarrow \) Jacobi relation

- Factorized \( \sigma(g, g, k) = (h, k) \)
- Group \( g \triangleleft HK \)
- \( H \cap K = k \)
- \( g, g = k \)

- Lattice \( \sigma(a, b) = \min(a, b, \max(0, b)) \)
- AND MANY MORE!!!

L. Vendramin 14: A "nice" set-theoretic solution \( \sigma \) to the YBE \( \rightarrow \) a shelf that captures important properties of \( \sigma \)
- e.g. the associated representations of the braid groups \( B_n \).

Carter-Elhamdadi-Saito 04, L.13: A cohomology theory for general YBE solutions

Applications:
- 
- 
- unifies basic (co)homology theories
- rack \& quandle
- group, Hochschild
- Chevalley-Eilenberg

- guides in developing (co)homology theories for new algebraic structures
- graphical calculi
- additional structures (cup product etc.)
- knot invariants
- universal enveloping group
- YBE solution \( (A, \sigma) \to \text{UEG}(A, \sigma) = \langle A \mid a b = b a \rangle \)

\[ \text{H}^n(A, \sigma) \cong \text{H}^n(\text{UEG}(A, \sigma), \mathbb{Z}) \]

H. Lee, M. Polyak, P. Zograf, Quantum Symmetrizer (2014)

\( Q \) is the quantum symmetrizer when \( Q^2 = \text{Id} \), coeff in \( \sigma \) (Farinati & Garcia-Galetto 16)

\( Q^2 = \sigma \) (L.14, proof: algebraic discrete Morse theory)

(c.f. examples A, C, D above + Young tableaux).