Knotted 3-Valent Graphs, Branched Braids, and Multiplication-Conjugation Relations in a Group

Victoria Lebed
Osaka City University Advanced Mathematical Institute

This survey is devoted to a new algebraic structure called ***qualgebra***. Our topological motivation is the study of *knotted 3-valent graphs* and closely related *branched braids* via combinatorially defined coloring invariants. From an algebraic viewpoint, our structure a part of an alternative *axiomatization of groups*, describing the properties of conjugation operation and its interactions with the group multiplication. Qualgebras can thus be metaphorically seen as a widening of the bridge between algebra and topology formed by the *quandle* structure, popular among knot theorists; see Table 1 to better understand how this bridge works.

Only a brief and rather informal exposition of different facets of qualgebras is given here. For more details, comments, and proofs, see [20, 15]. However, Sections 2.1, 2.3, 2.4, and 3.1 contain some recent unpublished results, which will be thoroughly treated elsewhere.

1 How a knot theorist would invent qualgebras

1.1 Quandles as an algebraization of knots

Diagram coloring techniques count among the most powerful combinatorial tools in Knot Theory. A famous example is given by Fox colorings, which are a particular case of quandle colorings. In this section we briefly recall the latter.

Take a set S and a binary operation $\lhd$ on it. An $(S,\lhd)$*-coloring* of a knot diagram D is an assignment of an element of S to each arc of D in such a way that the condition on Figure 1 Ⓐ (motivated below) is satisfied around each crossing. Unoriented arcs in our diagrams mean that the diagrams should be considered for all coherent orientations of such arcs.

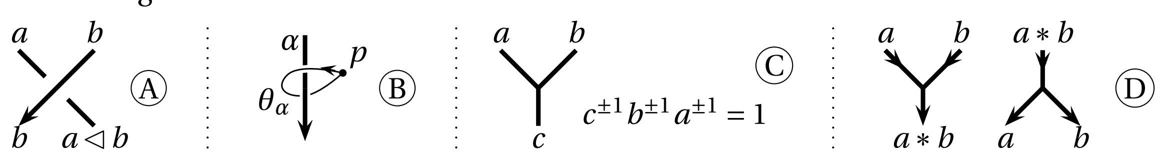

Figure 1: Coloring rules and their topological motivations

Now, we want colorings to say something about the knot K_D represented by D, independently of the diagram chosen. Therefore, we want Reidemeister moves (Figure 2) to induce only local coloring changes, keeping fixed all the colors outside the small ball where the move is realized. This happens if and only if operation $\lhd$ satisfies the following properties:

RIII	$\longleftrightarrow$	*self-distributivity*:	$(a\lhd b)\lhd c = (a\lhd c)\lhd(b\lhd c)$,	(Q_{SD})
RII	$\longleftrightarrow$	*invertibility*:	$\forall\, b,\ a\mapsto a\lhd b$ is invertible,	(Q_{Inv})
RI	$\longleftrightarrow$	*idempotence*:	$a\lhd a = a$.	(Q_{Idem})

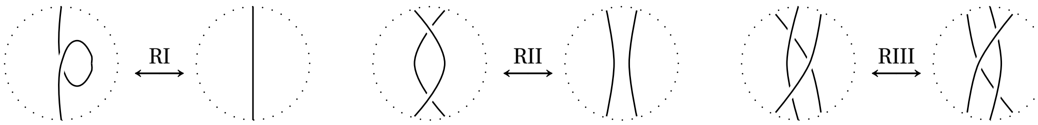

Figure 2: Reidemeister moves for knot diagrams

Data $(S, \lhd)$ satisfying (Q_{SD})-(Q_{Idem}) is called a ***quandle***. This structure has been actively studied since the pioneer 1982 papers of D. Joyce and S.V. Matveev [14, 23]. The argument above implies that the number of colorings of a knot diagram by a quandle is stable by Reidemeister moves, and thus defines an invariant of the underlying knot:

$$\boxed{\text{knot invariant} \overset{\text{colorings}}{\leftsquigarrow} \text{quandle}}$$

Such *quandle invariants* turn out to be extremely efficient in practice.

The central example of quandle is given by a group G and operation $g \lhd h = h^{-1}gh$ on it; it is the ***conjugation quandle*** of G, denoted by $\mathrm{Conj}(G)$. Now fix a diagram D of a knot K_D. Recall *Wirtinger presentation* of the knot group $\pi_1(\mathbb{R}^3 \setminus K_D)$, with one generator θ_α for each arc α of D, as shown on Figure 1B (point p is chosen in front of the diagram). Around each crossing, compare the relations imposed on the θ_α with the coloring rule from Figure 1A. One readily identifies $\mathrm{Conj}(G)$-colorings of D with representations of the knot group in G:

$$\boxed{\mathcal{Col}_{\mathrm{Conj}(G)}(D) \xleftrightarrow{\text{bijection}} \mathrm{Hom}(\pi_1(\mathbb{R}^3 \setminus K_D), G)}$$

Quandle invariants thus generalize the classical study of knot groups.

1.2 Extending quandle colorings to 3-graphs

Knotted 3-valent graphs (simply called *3-graphs* in what follows; cf. Figures 5 and 6 for typical examples) have recently attracted a lot of attention, among others due to applications to handle-body classification and to foams (a particular type of surfaces appearing in some categorification constructions and in 3-manifold studies). According to [19, 26, 27], the study of such graphs up to isotopy is equivalent to the study of their diagrams up to Reidemeister moves I-VI (Figures 2 and 3), opening the way to *combinatorial invariants*.

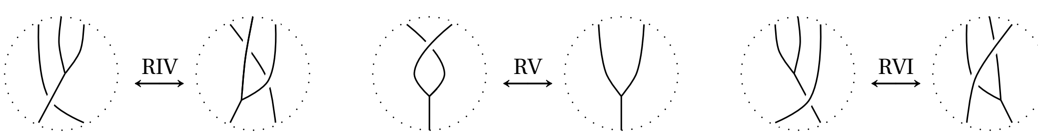

Figure 3: Additional Reidemeister moves for knotted 3-valent graph diagrams

A generalization of the (very powerful) quandle colorings to graphs is a possible source of such combinatorial invariants. The main challenge is to complete the coloring rule around crossings (Figure 1A) with a rule around trivalent vertices. Wirtinger presentation of the graph group suggests a solution when colors come from a conjugation quandle; it is given on Figure 1C, where the choices in $\pm$ depend on arc orientations. This idea was extended to more general quandles in [22, 11, 24, 12, 13]. In [20] we proposed an alternative solution, which consists in enriching the notion of quandle in a particular way.

Our method works for *well-oriented* 3-graphs — that is, having only zip and unzip vertices (Figure 4). Since every 3-graph is well-orientable, our method also allows to compare two unoriented 3-graphs by considering all their well-oriented versions.

Figure 4: Zip and unzip vertices for 3-graphs

Now, suppose our quandle $(S, \lhd)$ to be endowed with a second binary operation $*$, and use it to define a coloring rule around trivalent vertices as shown on Figure 1 D. As usual, one checks if this rule forces Reidemeister moves to induce only local changes in diagrams' colorings. It happens if and only if operations $\lhd$ and $*$ are compatible in the following sense:

$$\begin{array}{llrll}
\text{RIV} & \longleftrightarrow & \textit{translation composability}: & a \lhd (b * c) = (a \lhd b) \lhd c, & (QA_{Comp}) \\
\text{RVI} & \longleftrightarrow & \textit{distributivity}: & (a * b) \lhd c = (a \lhd c) * (b \lhd c), & (QA_D) \\
\text{RV} & \longleftrightarrow & \textit{semi-commutativity}: & a * b = b * (a \lhd b). & (QA_{Comm})
\end{array}$$

Data $(S, \lhd, *)$ satisfying (Q_{SD})-(Q_{Idem}) and (QA_{Comp})-(QA_{Comm}) is called a ***qualgebra***. This term consists of words "quandle" and "algebra" zipped together, which underlines the presence and the importance of two different operations in the story. Note that axiom (Q_{SD}) can be omitted since it follows from (QA_{Comp}) and (QA_{Comm}). Our choice of axioms guarantees that the number of colorings of a graph diagram D by a qualgebra is stable by Reidemeister moves, and thus defines an invariant of the underlying well-oriented 3-graph Γ_D:

$$\boxed{\text{3-graph invariant} \overset{\text{colorings}}{\leftsquigarrow} \text{qualgebra}}$$

The central example of qualgebra is given, once again, by a group G, with conjugation and multiplication operations: $g \lhd h = h^{-1}gh$, $g * h = gh$. It is the ***group qualgebra*** of G, denoted by $QA(G)$. The coloring rule from Figure 1 D recovers in this case the one from Figure 1 C, prescribed by Wirtinger presentation of the graph group. One that gets

$$\boxed{\mathcal{C}ol_{QA(G)}(D) \overset{\text{bijection}}{\longleftrightarrow} \operatorname{Hom}(\pi_1(\mathbb{R}^3 \backslash \Gamma_D), G)}$$

We finish this section with a computation example. Here instead of counting all colorings of a diagram by a qualgebra S, we restrict ourselves to *isosceles colorings*. This means that both incoming (or outcoming) edges of any zip (respectively, unzip) vertex are colored by the same element of S; in other words, one imposes $a = b$ in Figure 1 D. Reidemeister moves do not change the property of being isosceles, hence the number of isosceles colorings $\#\mathcal{C}ol_S^{iso}(D)$ is a graph invariant. The 3-graphs we are interested in are standard and Kinoshita-Terasaka Θ-curves, with diagrams given on Figure 5. An isosceles coloring of Θ_{st} is entirely determined by the choice of $x \in S$, so $\#\mathcal{C}ol_S^{iso}(\Theta_{st}) = \#S$. Any other well-orientation of Θ_{st} leads to the same result. For Θ_{KT}, the choice of $x, y \in S$ determines everything, but this choice is not free, since a, b and c can be expressed in terms of x and y in different ways:

$$(\star)\begin{cases} a &= x \lhd (y * y) = y \lhd x, \\ b &= x \widetilde{\lhd} y = y \widetilde{\lhd} (x * x), \\ c &= (y * y) \lhd x = (x * x) \widetilde{\lhd} y. \end{cases}$$

Here we use notion $x \widetilde{\lhd} y$, classical in Quandle Theory: it stands for the unique $z \in S$ satisfying $z \lhd y = x$ (cf. axiom (Q_{Inv})). Now, for any x, the choice $y = x$ provides a solution to ($\star$), so $\#\mathcal{Col}_S^{iso}(\Theta_{KT}) \geqslant \#S = \#\mathcal{Col}_S^{iso}(\Theta_{st})$. To separate these quantities (and thus to distinguish the two Θ-curves), take as S the group qualgebra of the symmetric group S_4. One checks that $x = (123)$ and $y = (432) \neq x$ satisfy ($\star$), giving $\#\mathcal{Col}_{QA(S_4)}^{iso}(\Theta_{KT}) > \#QA(S_4) = \#\mathcal{Col}_{QA(S_4)}^{iso}(\Theta_{st})$.

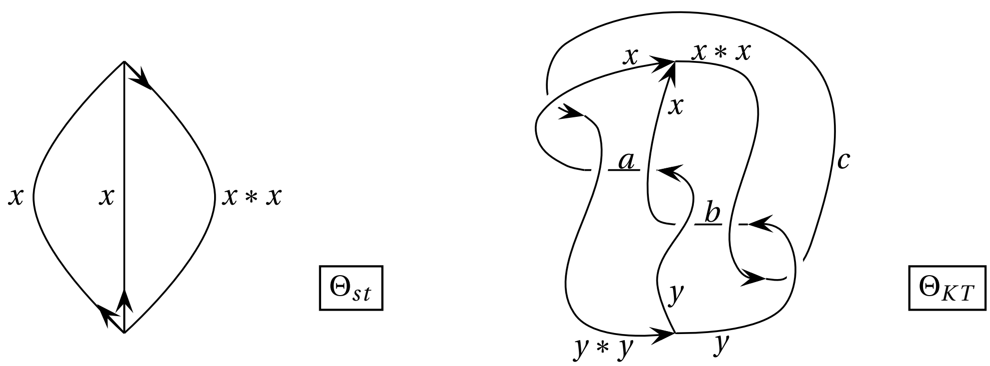

Figure 5: Isosceles colorings for diagrams of standard and Kinoshita-Terasaka Θ-curves

2 How an algebraist would invent qualgebras

2.1 An abstraction of the conjugation-multiplication interaction in a group

Let us return to quandles once again. Besides Knot Theory, they appear in another setting, completely algebraic this time. We saw above that conjugation operation defines a quandle structure on a group, and thus satisfies axioms (Q_{SD})-(Q_{Idem}). In fact, one can say more: if a property involving only conjugation holds true in every group, then it is a consequence of these three axioms. The reason lies in the structure of the free quandle on a set X, which can be seen inside the free group on X. Quandle thus provide an *axiomatization of conjugation*.

In a similar way, conjugation and multiplication operations define a qualgebra structure on a group, and thus satisfy all qualgebra axioms. Moreover, axioms (QA_{Comp})-(QA_{Comm}) capture all essential relations between conjugation and multiplication (cf. Table 1). However, formalizing this idea is not so easy. For instance, relation

$$(b \lhd a) * (a \lhd b) = ((a \widetilde{\lhd} b) \lhd a) * b$$

holds in any group qualgebra (both sides equal $a^{-1}bab^{-1}ab$), but fails in the free qualgebra on two elements — and thus does not follow from qualgebra axioms.

abstract level	quandle axioms	specific qualgebra axioms
topology	moves RI-RIII	moves RIV-RVI
groups	conjugation	conjugation-multiplication interaction

Table 1: Different viewpoints on quandles and qualgebras

The remainder of this section is devoted to various examples of qualgebras. Algebraic properties of some of them are very different from those of groups. This confirms that the interest of qualgebras goes beyond the realm of groups. One more conclusion is that conjugation and multiplication operations do not suffice for an *alternative axiomatization of groups*; the missing ingredients will be determined in Section 2.3.

- The first example is still close to groups. Consider sub-qualgebras of a group qualgebra $QA(G)$ — that is, subsets stable by conjugation and multiplication. If G is finite, one gets only subgroups of G. For infinite G new examples appear: for instance, the sub-qualgebra $\mathbb{N}$ of $QA(\mathbb{Z},+)$ contains no inverses, and thus is not a group qualgebra.
- Now, consider qualgebras $(S,\lhd,*)$ with $a\lhd b = a$, called ***trivial qualgebras***. In this case, the only condition imposed on $*$ by axioms (QA_{Comp})-(QA_{Comm}) is the commutativity. Colorings by trivial qualgebras do not distinguish over-crossings from under-crossings, and thus do not capture the knottedness of 3-graphs. Such qualgebras thus yield only abstract graph invariants.
- A more sophisticated example can be constructed as follows. Take a set X equipped with a commutative operation $\star$ and a distinguished zero element 0 (this means that $0\star x = x\star 0 = 0$ for all x). Fix an $n \in \mathbb{N}$. Extend operation $\star$ to $X^{\times n}$ coordinate-wise, and denote by $\cdot$ the usual right action of the symmetric group S_n on $X^{\times n}$. Now, the set

  $$Q_{X,n} = \{\, ((x_1,\dots,x_n),g) \in X^{\times n}\times S_n \,|\, x_i = x_j = 0 \text{ whenever } g(i) = j \text{ with } i \neq j \,\}$$

  can be endowed with the following qualgebra structure:

  $$(\overline{x},g)\lhd(\overline{y},h) = (\overline{x}\cdot h, h^{-1}gh),$$
  $$(\overline{x},g)*(\overline{y},h) = (\overline{x}\star\overline{y}, gh).$$

  Consider the simplest example $X_2 = \{0,a\}$. Then

  $$Q_{X_2,2} = \{\, ((x_1,x_2),\mathrm{Id}) \,|\, x_1,x_2 \in X_2 \,\} \coprod \{\, ((0,0),\tau) \,\}$$

  (where τ is the non-trivial element of S_2) consists of five elements. Two operations $\star_1$ and $\star_2$ can be fed into our machine: 0 is a zero element for both, and we have $a\star_1 a = 0$ and $a\star_2 a = a$. The two resulting qualgebras are non-isomorphic. Their operations $*_i$ are commutative, associative, but non-cancellative:

  $$((0,0),\tau)*_i((0,a),\mathrm{Id}) = ((0,0),\tau)*_i((a,0),\mathrm{Id}) = ((0,0),\tau), \qquad i = 1,2.$$

2.2 Towards a classification of qualgebras: the 4-element case

Up to size 3, all qualgebras are trivial. Things change in size 4. In [20] we classified all non-trivial 4-element qualgebras up to qualgebra isomorphism. Here we describe all the 9 isomorphism classes. On the set $Q = \{\, p,q,r,s \,\}$, consider the involution exchanging p and q:

$$\overline{p} = q, \overline{q} = p, \overline{r} = r, \overline{s} = s.$$

Put $x\lhd r = \overline{x}$, and $x\lhd y = x$ for other y. As for the second operation, take the commutative operation $*$ defined as follows:

- $\overline{x} * \overline{y} = \overline{x * y}$ for all $x, y \in Q$;
- r enjoys the absorption property: $r * x = r$ for all $x \neq r$;
- one has $r * r = s * s = p * q = s$;
- $q * q$ and $q * s$ are any elements chosen in $\{p, q, s\}$.

The alternatives in the last point lead to $3 \times 3 = 9$ pairwise non-isomorphic structures.

The absorption property for r prevents these qualgebras from being cancellative with respect to $*$ and, a fortiori, from embedding into a group. Further, out of these nine structures, precisely two are associative. They are in fact the sub-qualgebras of $Q_{X_2,2}$ from Section 2.1 obtained by omitting the element $((a,a),\mathrm{Id})$; the two possible operations $\star_1$ and $\star_2$ give non-isomorphic structures. Lastly, three qualgebras out of the nine have neutral elements, and none are unital associative. Thus even in this small size qualgebras can exhibit a wide range of algebraic behavior, confirming the interest of this structure.

To illustrate topological applications of 4-element qualgebras, consider the diagrams of standard and Hopf cuff graphs depicted on Figure 6. Analyzing the colors around trivalent vertices, one gets for any qualgebra S the following bijections:

$$\mathcal{C}ol_S(C_{st}) \overset{bij}{\longleftrightarrow} \{(a,b,c) \in S^{\times 3} \,|\, b * a = a,\ b * c = c\},$$
$$\mathcal{C}ol_S(C_H) \overset{bij}{\longleftrightarrow} \{(a,b,c) \in S^{\times 3} \,|\, b * a = (a \widetilde{\lhd} c) \lhd a,\ b * c = c \lhd a\}.$$

For a trivial qualgebra S, these sets coincide. However, for the non-trivial 4-element qualgebra Q above with $q * q = s$ and $q * s = q$, one gets $\#\mathcal{C}ol_Q(C_{st}) = 18$ (and the same value for any well-orientation of C_{st}, due to the commutativity of $*$), and $\#\mathcal{C}ol_Q(C_H) = 14$. This distinguishes the two cuff graphs.

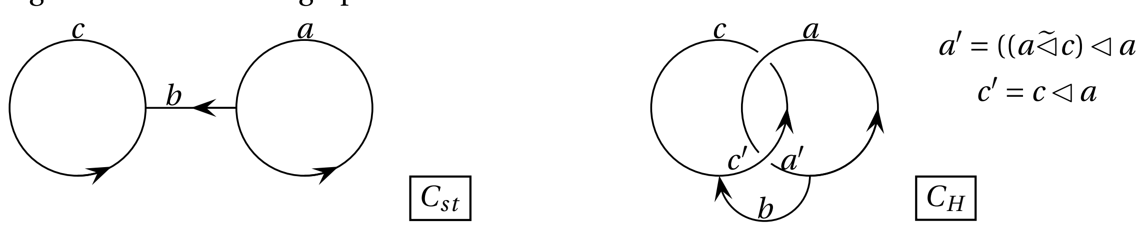

Figure 6: Qualgebra colorings for diagrams of standard and Hopf cuff graphs

2.3 Getting closer to groups: symmetric qualgebras

We now turn to distinctions between the notions of group and qualgebra. Above were given examples of qualgebras which are not associative and/or not cancellative. Here these two properties will be shown to be essentially the only ones needed for a qualgebra to be a group.

The notion of *symmetric quandle*, introduced in 1996 by S. Kamada ([17]), should first be recalled. It is a quandle $(S, \lhd)$ endowed with an involution $\rho : S \to S$ (called a ***good involution***), compatible with operation $\lhd$ in the following sense:

$$\rho(a) \lhd b = \rho(a \lhd b), \tag{1}$$
$$a \lhd \rho(b) = a \widetilde{\lhd} b. \tag{2}$$

The topological role of a good involution is to render quandle invariants independent of orientations. Concretely, if $(S, \lhd, \rho)$ is a symmetric quandle, then a bijection $\mathcal{C}ol_S(D) \leftrightarrow \mathcal{C}ol_S(-D)$, where diagrams D and $-D$ differ by the orientation only, can be given by the rule $a\downarrow \longleftrightarrow \uparrow\rho(a)$. Now, we want the same kind of rule to induce a bijection between the $(S, \lhd, *)$-coloring sets of 3-graph diagrams which differ by the orientation of some edges only (all the graphs involved are supposed well-oriented). For this to hold, ρ should be a good involution for the quandle $(S, \lhd)$, compatible with $*$ in the following sense:

$$(a * b) * \rho(b) = \rho(b) * (b * a) = a. \tag{3}$$

The resulting structure $(S, \lhd, *, \rho)$ is called a ***symmetric qualgebra***.

As one would expect, the central example is given by group qualgebras, for which the inversion $\rho(g) = g^{-1}$ defines a good involution. Table 1 can now be continued with Table 2.

abstract level	good involution axioms
topology	unoriented 3-graphs
groups	conjugation-inversion and multiplication-inversion interactions

Table 2: Different viewpoints on symmetric qualgebras

In a symmetric qualgebra, maps $a \mapsto a * b$ and $a \mapsto b * a$ are bijections for all b, according to axiom (3). Consequently,

- for any b, property (3) defines $\rho(b)$ uniquely; good involutions can thus be safely omitted from the description of a symmetric qualgebra;
- the multiplication table for $*$ is a Latin square (i.e., every element occurs exactly once in each column and in each row).

Using these observations, symmetric trivial qualgebras are particularly easy to describe. They correspond to Latin squares which

1. are symmetric with respect to the main diagonal, and
2. together with a row corresponding to a permutation σ necessarily contain a row corresponding to σ^{-1} (the two rows can coincide).

Let us now turn to examples.

- Among 3-element qualgebras (which are necessarily trivial), there are precisely 3 symmetric ones, as usual up to symmetric qualgebra isomorphism:

$*$	x	y	z
x	x	y	z
y	y	z	x
z	z	x	y
ρ	x	z	y

$QA(\mathbb{Z}/3\mathbb{Z})$

$*$	x	y	z
x	x	z	y
y	z	y	x
z	y	x	z
ρ	x	y	z

$*$	x	y	z
x	y	x	z
y	x	z	y
z	z	y	x
ρ	x	y	z

not groups

- Among trivial 4-element qualgebras, there are precisely 4 symmetric ones:

$*$	x	y	z	w
x	x	y	z	w
y	y	z	w	x
z	z	w	x	y
w	w	x	y	z
ρ	x	w	z	y

$QA(\mathbb{Z}/4\mathbb{Z})$

$*$	x	y	z	w
x	x	y	z	w
y	y	x	w	z
z	z	w	x	y
w	w	z	y	x
ρ	x	y	z	w

$QA(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$

$*$	x	y	z	w
x	z	y	x	w
y	y	z	w	x
z	x	w	z	y
w	w	x	y	z
ρ	x	w	z	y

$*$	x	y	z	w
x	x	y	w	z
y	y	x	z	w
z	w	z	y	x
w	z	w	x	y
ρ	x	y	z	w

not groups

- Non-trivial 4-element qualgebras are not cancellative and thus not symmetric.

Even though good involutions bring the structure of qualgebra closer to that of group, the examples above show that symmetric qualgebra stay more general than groups. The missing property turns out to be the associativity: group qualgebras are precisely symmetric qualgebras which are *associative* (i.e., their operation $*$ is associative); see Figure 7.

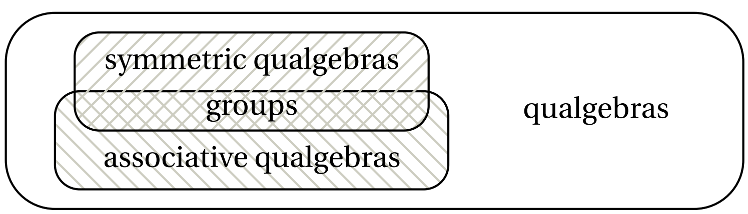

Figure 7: Qualgebras versus groups

In particular, this allows to deduce the non-associativity of two 3-element and two 4-element symmetric qualgebras above from the absence of neutral elements for their operations $*$ (and thus their failure to be group qualgebras), which is much easier to check.

2.4 From quandles to qualgebras

Above we analyzed how far the notion of qualgebra is from that of group. A comparison of the notions of qualgebra and quandle will be given here.

Let us first discuss when a quandle $(S, \lhd)$ is *qualgebraizable* — that is, admits a second operation $*$ turning it into a qualgebra. For this, consider *right translations* $T_b : a \mapsto a \lhd b$, written here on the right of their arguments. Axioms (Q_{SD})-(Q_{Idem}) imply that

1'. every T_b is an automorphism of the quandle $(S, \lhd)$ fixing b;

2'. the map $T : b \mapsto T_b$ is a quandle morphism from $(S, \lhd)$ to Conj(Aut(S)) — that is, one has $T_{b \lhd c} = T_c^{-1} T_b T_c$;

3'. the image of T is a sub-quandle of Conj(Aut(S)).

Now, if $(S, \lhd, *)$ is a qualgebra, then in addition

1. maps T_b are automorphisms of the qualgebra $(S, \lhd, *)$;
2. the map $T : b \mapsto T_b$ is a qualgebra morphism from $(S, \lhd, *)$ to QA(Aut(S)) — that is, one has $T_{b*c} = T_b T_c$;
3. $T(S)$ is a sub-qualgebra of QA(Aut(S)), and is in particular stable under composition;
4. the restriction of T_b to the sub-qualgebra of S generated by b is the identity map.

Property 3 is an important *necessary qualgebraizability condition*, which is unfortunately not sufficient (a counter-example is given below). Neither does it give estimations for the number of qualgebraizations of a given quandle: the related property 2 determines $b * c$ only modulo Ker(T), which can be very large. As for now, no satisfying qualgebraizability criterion is known to the author.

We now give some examples where qualgebraizations are unique, are numerous, or do not exist at all.

- As shown above, the qualgebraizations of a trivial quandle are given by commutative operations $*$. For the trivial quandle with n elements, this gives $n^{\frac{n(n+1)}{2}}$ qualgebraizations. However, counting these qualgebraizations up to qualgebra isomorphism is much more difficult. For instance, for $n = 2$ these 8 structures fall into 4 equivalence classes, and for $n = 3$ the 729 structures form 129 classes.
- Consider an *Alexander quandle* $(M, a \lhd b = \alpha a + (1 - \alpha)b)$, where M is a module over a ring R, and α is a fixed invertible element from R. One calculates
$$(a)T_b T_c = (a \lhd b) \lhd c = \alpha^2 a + \alpha(1-\alpha)b + (1-\alpha)c.$$
Our quandle is qualgebraizable only if $T_b T_c$ equals $T_d : a \mapsto \alpha a + (1-\alpha)d$ for some $d \in M$. But this would imply that the value of $\alpha^2 a - \alpha a$ does not depend on a. Since α is invertible, the value of $\alpha a - a$ is also a constant, and thus $a \lhd b = \alpha a + b - \alpha b = a$. One concludes that among Alexander quandles, only the trivial ones are qualgebraizable.
- There are 3 quandles of size 3:
  - The trivial one was shown to admit 4 qualgebraizations.
  - The Alexander quandle $(\mathbb{Z}/3\mathbb{Z}, a \lhd b = 2b - a)$ (the colorings by which are precisely the famous Fox colorings) was proved not to be qualgebraizable.
  - The sub-quandle $Q' = \{p, q, r\}$ of the quandle Q from Section 2.2 satisfies necessary algebraizability condition 3 above, since $T(Q')$ is a 2-element subgroup (hence sub-qualgebra) of Aut(Q'). However, Q' is not qualgebraizable. Indeed, according to property 4 above, element $r * r$ should be fixed by T_r, implying $r * r = r$; but this contradicts property 2, which gives $T_{r*r} = T_r T_r = \mathrm{Id} \neq T_r$.

- The quandle Q from Section 2.2 admits 9 qualgebraizations (up to isomorphism). Note that above we showed its sub-quandle $\{p,q,r\}$ not to be qualgebraizable.
- The group qualgebra $QA(S_n)$ is a qualgebraization of the conjugation quandle $\mathrm{Conj}(S_n)$. This qualgebraization is unique for $n \geqslant 3$, since in this case the map T is injective.

3 Variations of qualgebra ideas

3.1 Towards qualgebra cohomology

Fix a qualgebra $(S, \lhd, *)$. In Section 1.2, we saw that any Reidemeister move induces a bijection between the sets $\mathcal{C}ol_S(D)$ and $\mathcal{C}ol_S(D')$ of $(S,\lhd,*)$-colorings of the two well-oriented 3-graph diagrams involved. The conclusion was that the cardinality $\#\mathcal{C}ol_S(D)$ of such a set is a 3-graph invariant. However, a lot of information is lost when passing from the coloring set to its cardinality. Here we show how to retrieve some of it, imitating what was done for quandle colorings of knots by Carter-Jelsovsky-Kamada-Langford-Saito in 1999 (cf. the original papers [1, 2], a very pedagogic survey [16], and numerous related publications).

The basic idea is to associate to every S-coloring $\mathscr{C}$ of a diagram D a quantity invariant under Reidemeister moves. Developing the approach of [1], we look for quantities of a particular form. Take two maps $\chi, \lambda : S \times S \to \mathbb{Z}$, evaluate them on all the crossings and trivalent vertices of D colored according to $\mathscr{C}$, as shown on Figure 8, and sum up the values obtained. The result is called the (χ,λ)***-weight*** of $\mathscr{C}$, denoted by $\omega_{\chi,\lambda}(\mathscr{C})$.

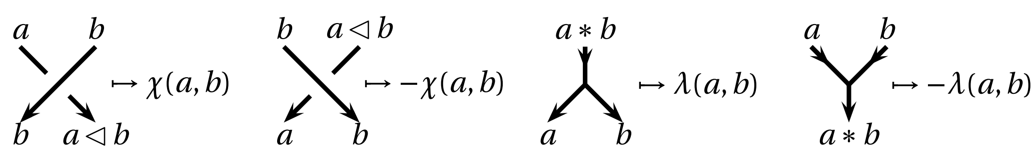

Figure 8: Qualgebra 2-cocycle $\rightsquigarrow$ weight

The invariance of the weight $\omega_{\chi,\lambda}(\mathscr{C})$ under Reidemeister moves is equivalent to the following relations for χ and λ:

$$\begin{aligned}
&\text{RIV} &&\longleftrightarrow && \chi(a, b*c) = \chi(a,b) + \chi(a \lhd b, c), && (4)\\
&\text{RVI} &&\longleftrightarrow && \chi(a*b, c) + \lambda(a \lhd c, b \lhd c) = \chi(a,c) + \chi(b,c) + \lambda(a,b), && (5)\\
&\text{RV} &&\longleftrightarrow && \chi(a,b) + \lambda(a,b) = \lambda(b, a \lhd b). && (6)
\end{aligned}$$

The relations for the remaining moves follow from the presented ones and are omitted.

A pair of maps $\chi, \lambda : S \times S \to \mathbb{Z}$ satisfying (4)-(6) is called a ***qualgebra 2-cocycle*** for S. As shown above, for such a pair the multi-set of weights $\{\omega_{\chi,\lambda}(\mathscr{C}) \mid \mathscr{C} \in \mathcal{C}ol_S(D)\}$ defines an invariant of the underlying well-oriented 3-graph Γ_D:

$$\boxed{\text{3-graph invariant} \underset{\text{\& weights}}{\overset{\text{colorings}}{\leftsquigarrow}} \text{qualgebra \& 2-cocycle}}$$

The same qualgebra thus gives rise to a whole family of so-called *cocycle invariants*. In particular, one recovers the qualgebra invariants from Section 1.2 when taking zero maps χ and λ.

The term "qualgebra 2-cocycle" was chosen to stress the analogy with *quandle 2-cocycles* from [1], which are indeed 2-cocycles for the celebrated *quandle cohomology* theory. As for now, no ***qualgebra cohomology*** theory is known. Topological arguments suggest what it should look like in small degrees, but its continuation to higher degrees remains mysterious. The general braided cohomology theory from [21] yields a cohomology theory for *rigid qualgebras* (with axiom (QA_{Comm}) omitted from the definition); topologically, these correspond to *rigid 3-graphs* (for which graph vertices are viewed as disks, not as points, excluding Reidemeister move V). However, this approach does not work for general qualgebras.

Let us describe some properties of 2-cocycles for our qualgebra S. They form an Abelian group $Z^2(S)$ under point-wise coordinate-wise addition. A subgroup $B^2(S)$ is formed by ***qualgebra 2-coboundaries*** — that is, 2-cocycles built out of maps $\phi : S \to \mathbb{Z}$ as follows:

$$\chi(a,b) = \phi(a \lhd b) - \phi(a),$$
$$\lambda(a,b) = \phi(a) + \phi(b) - \phi(a * b).$$

Such 2-cocycles are useless for distinguishing graphs, giving zero weights only. The quotient $H^2(S) = Z^2(S)/B^2(S)$ is a natural candidate for the title "degree 2 cohomology of S".

In order to show that the definitions from this section are not empty, we present computations for the 4-element qualgebras from Section 2.2. All the 9 qualgebras described there exhibit the same homological behavior. Namely, they satisfy

$$Z^2(Q) \cong \mathbb{Z}^8, \qquad B^2(Q) \cong \mathbb{Z}^4, \qquad H^2(Q) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}^4.$$

The torsion appearing in the quotient is particularly interesting.

We finish with two directions continuing the "color-and-weight" ideas.

1. Together with diagram arcs, one can color diagram regions with elements of our qualgebra (or of a more general qualgebra module). The philosophy of weights then naturally leads to a notion of *qualgebra 3-cocycles*, and to a generalizations of *shadow cocycle invariants*, constructed in the case of quandles in [17, 3].

2. The evaluation rules for trivalent vertices from Figure 8 are the simplest ones making things work. One can add a third map $\curlyvee : S \times S \to \mathbb{Z}$ to the initial data, use it for evaluations on zip vertices, and write down the compatibility conditions for $\curlyvee$, λ and χ imposed by Reidemeister moves. This could lead to a richer family of 3-graph invariants.

3.2 Weak qualgebras and branched braids

Many combinatorial knot invariants directly generalize to links, braids, tangles and other 1-dimensional topological objects. In the case of braids one can often obtain even stronger results, since some flexibility is gained by excluding Reidemeister move I from the story. For example, when extending quandle invariants to braids, one gets two enhancements for free:

1. a weaker structure called *rack* (= data $(S, \lhd)$ satisfying (Q_{SD})-(Q_{Inv}) only) can serve as a coloring set;

2. the S-colors of the n upper arcs of a braid β with n strands uniquely determine the colors of all remaining arcs, in particular of the n lower arcs; this defines a map $B_\beta : S^{\times n} \to S^{\times n}$, which is a braid invariant.

In the opposite direction, Alexander and Markov theorems present knots as certain equivalence classes of braids, via the closure operation. Hence braid invariants provide a potential source of knot invariants. In this section we introduce a topological notion which plays for 3-graphs the same role as braids play for knots, and present a weak version of qualgebras sufficient for producing invariants of these new objects.

The *closure* map for braids is recalled on Figure 9. Alexander theorem asserts its surjectivity by presenting every link as the closure of some braid. Markov theorem describes its kernel by showing that any two braids with isotopic closures are connected by a finite sequence of Reidemeister moves II-III and *Markov moves* 1-2 (see Figure 10; thick lines here replace an arbitrary number of strands).

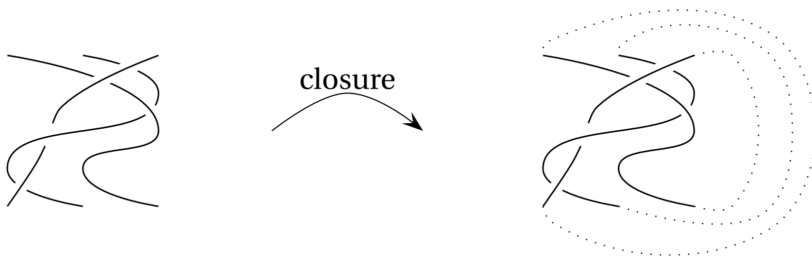

Figure 9: Braid $\rightsquigarrow$ link

Figure 10: Markov moves

When studying 3-graphs, braids should be replaced with ***branched braids***. These are knotted graphs in $\mathbb{R}^2 \times [0,1]$ with n univalent vertices on the top, m univalent vertices on the bottom, some trivalent vertices in between, and no cups or caps (with respect to the third coordinate projection $\mathbb{R}^2 \times [0,1] \to [0,1]$). The closure operation is still defined for branched braids with $n = m$, as shown on Figure 11.

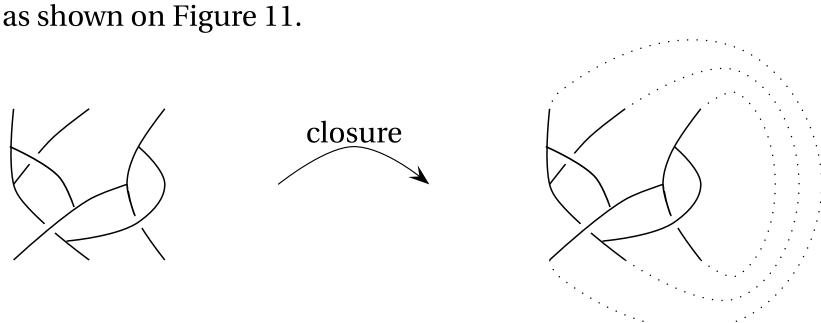

Figure 11: Branched braid $\rightsquigarrow$ 3-graph

K. Kanno and K. Taniyama ([18]) proved that all 3-graphs are obtained this way, giving an Alexander-type theorem for branched braids; see also [25] for a related result for theta-curves. A Markov-type theorem for branched braids was established by S. Kamada and the author ([15]): we showed any two branched braids with isotopic closures to be connected by a finite sequence of Reidemeister moves II-VI and Markov moves 1-2. This result generalizes to *graph-braids* (containing vertices of arbitrary valence), and to virtual and welded settings.

closure map	usual braids	branched braids
surjectivity	Alexander, 1923	Kanno-Taniyama, 2010
kernel	Markov, 1935	Kamada-L., 2014

Table 3: Alexander- and Markov-type theorems in different settings

On the level of invariants, the two theorems imply that a branched braid invariant stable under Markov moves automatically gives rise to a 3-graph invariant.

In the opposite direction, qualgebra colorings work well for branched braids. Among the two enhancements mentioned above for quandle colorings of braids, only the first one adapts to this setting. Indeed, a ***weak qualgebra*** (= data $(S, \lhd, *)$ satisfying (Q_{SD})-(Q_{Inv}) and (QA_{Comp})-(QA_{Comm}) only) can serve as a coloring set for branched braid diagrams:

$$\text{branched braid invariant} \overset{\text{colorings}}{\leftsquigarrow} \text{weak qualgebra}$$

However, contrary to the case of usual braids, here upper colors do not determine lower colors* because of unzip vertices: the knowledge of $a * b$ does not give you a and b. Hence one has to content oneself with counting (weak) qualgebra colorings, possibly with weights.

3.3 Qualgebras in Set Theory

Besides the topological and algebraic settings described above, axioms (QA_{Comp})-(QA_{Comm}) also emerge in a completely different set-theoretical context. Namely, together with the associativity of $*$ and the existence of a neutral element 1 for $*$ satisfying moreover $1 \lhd a = 1$ and $a \lhd 1 = a$ for all a, they define a *(right-)distributive monoid* (or, in other sources, RD algebra). Examples include elementary embeddings, Laver tables, and extended braids. All of them admit rich distributive monoid structures, motivating an extensive study of the concept (see for instance [4, 9, 10, 5], or Chapter XI of [6] for a comprehensive exposition). A weaker *augmented (right-)distributive system* structure of P. Dehornoy obeys only axioms (Q_{SD}), (QA_{Comp}), and (QA_D). The major example here is that of parenthesized braids ([7, 8]). Our qualgebras are particular cases of augmented distributive systems.

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Osaka City University Advanced Mathematical Institute
Osaka City University
Osaka 558-8585
JAPAN
E-mail address: `lebed.victoria@gmail.com`