Laver Tables: from Set Theory to Braid Theory

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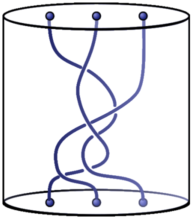
$0, 1, 2, 3, \ldots;$
$\aleph_0, \aleph_1, \aleph_2, \ldots;$
$\aleph_\omega, \ldots$

A_3	1	2	3	4	5	6	7	8
1	2	4	6	8	2	4	6	8
2	3	4	7	8	3	4	7	8
3	4	8	4	8	4	8	4	8
4	5	6	7	8	5	6	7	8
5	6	8	6	8	6	8	6	8
6	7	8	7	8	7	8	7	8
7	8	8	8	8	8	8	8	8
8	1	2	3	4	5	6	7	8



Part 1

A Laver table is...

Basic definitions

A shelf (= self-distributive structure)

is a set S with an operation $\triangleright$ satisfying

$$\boxed{a \triangleright (b \triangleright c) = (a \triangleright b) \triangleright (a \triangleright c)} \qquad \text{(SD)}$$

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Theorem (Laver, '95): properties (SD) and (Init) uniquely define $\triangleright$.

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⚠ False for $\{1, 2, 3, \ldots, q\}$ with $q \neq 2^n$.

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$$\gamma \qquad (\gamma \triangleright \gamma) \triangleright \gamma$$

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$$\gamma = 1 \qquad (\gamma \triangleright \gamma) \triangleright \gamma = 3$$

$$\gamma \triangleright \gamma = 2 \qquad ((\gamma \triangleright \gamma) \triangleright \gamma) \triangleright \gamma = 4 \qquad \ldots$$

Laver tables in Set Theory

$$\begin{tikzcd} & \textbf{Set Theory} \arrow[dl, dotted] \arrow[dr, dotted] & \\ \text{Laver tables} \arrow[rr, dotted, "\text{approximation}"] & & \text{Free shelf } \mathcal{F}_1 \end{tikzcd}$$

Richard Laver

Laver tables in Set Theory: details

"Super-infinite" sets

Finite $\Longleftrightarrow$ every **self-embedding** is bijective.

Infinite $\Longleftrightarrow$ admits a non-bijective self-embedding.

Example: $\mathbb{N}$ is infinite ($n \mapsto n+1$ is a non-bijective self-embedding),

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Example: $\mathbb{N}$ is infinite ($n \mapsto n+1$ is a non-bijective self-embedding), but not super-infinite (f is elementary $\Rightarrow f(1) = 1$, $f(n+1) = f(n) + 1$).

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Axiom I3

V_λ (a certain limit rank) is super-infinite.

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⚠ I3 can neither be proved nor refuted in Zermelo-Fraenkel system.

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Self-embeddings

Set S $\quad\rightsquigarrow\quad$ $\mathrm{Emb}(S) := \{\, f : S \hookrightarrow S \,\}$ $\quad\rightsquigarrow\quad$ a shelf $(\mathrm{Emb}(S), \triangleright)$

$$f \triangleright g = \begin{cases} fgf^{-1} & \text{on the image } \mathrm{Im}(f) \text{ of } f, \\ \mathrm{Id} & \text{on the complement of } \mathrm{Im}(f). \end{cases}$$

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V_λ is super-infinite (= admits a non-bij. elementary self-embedding f_0).

Laver, 90's:

I3 $\Rightarrow$ ❁ f_0 generates a sub-shelf $F \subseteq \mathrm{Emb}(V_\lambda)$, with $F \cong \mathcal{F}_1$;

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- F has quotients of size 2^n $\leftsquigarrow$ Laver tables!
- $\boxed{\varprojlim_{n \in \mathbb{N}} A_n \supset \mathcal{F}_1}$ $\leftsquigarrow$ A_n are finite approximations of $\mathcal{F}_1$

Going beyond Set Theory?

Elementary definition

$A_n = (\{ 1, 2, 3, \ldots, 2^n \}, \triangleright)$ satisfying

$$\boxed{a \triangleright (b \triangleright c) = (a \triangleright b) \triangleright (a \triangleright c)} \tag{SD}$$

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Elementary properties

- A projective system of shelves:

$$\begin{aligned} p_n \ : \ A_n &\longrightarrow A_{n-1}, \\ a &\longmapsto a \mod 2^{n-1}. \end{aligned}$$

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- Periodic rows:

$p \triangleright 1$	$<$	$p \triangleright 2$	$<$	$\cdots$	$<$	$p \triangleright 2^r$	... periodic repetition ...
$= p + 1$						$= 2^n$	

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$$\begin{array}{|ccccccc|c} \hline \underset{= p+1}{p \triangleright 1} & < & p \triangleright 2 & < & \cdots & < & \underset{= 2^n}{p \triangleright 2^r} & \ldots \text{ periodic repetition } \ldots \\ \hline \end{array}$$

$\pi_n(p) := 2^r$ is the **period** of p in A_n.

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$$\boxed{A_n \cong \mathcal{F}_1 / (\cdots((\gamma \triangleright \gamma) \triangleright \gamma) \cdots) \triangleright \gamma = \gamma}$$

$$1 \leftrightarrow \gamma$$

$$2 \leftrightarrow \gamma \triangleright \gamma$$

$$3 \leftrightarrow (\gamma \triangleright \gamma) \triangleright \gamma$$

$$4 \leftrightarrow ((\gamma \triangleright \gamma) \triangleright \gamma) \triangleright \gamma \qquad \ldots$$

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- Some "nice" rows and columns

A_n	1	2	3	$\cdot$	2^{n-1}	$\cdot$	2^n	π_n
		$\cdots$				$\cdots$		
2^{n-1}	$2^{n-1}+1$	$2^{n-1}+2$	$2^{n-1}+3$	$\cdots$	2^n	$\cdots$	2^n	2^{n-1}
		$\cdots$				$\cdots$		
2^n-3	2^n-2	2^n	2^n-2	$\cdots$	2^n	$\cdots$	2^n	2
2^n-2	2^n-1	2^n	2^n-1	$\cdots$	2^n	$\cdots$	2^n	2
2^n-1	2^n	2^n	2^n	$\cdots$	2^n	$\cdots$	2^n	1
2^n	1	2	3	$\cdots$	2^{n-1}	$\cdots$	2^n	2^n

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2^n-3	2^n-2	2^n	2^n-2	$\cdots$	2^n	$\cdots$	2^n	2
2^n-2	2^n-1	2^n	2^n-1	$\cdots$	2^n	$\cdots$	2^n	2
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⚠ Theorems under Axiom I3!

A_0	1
1	1

A_1	1	2
1	2	2
2	1	2

A_2	1	2	3	4
1	2	4	2	4
2	3	4	3	4
3	4	4	4	4
4	1	2	3	4

A_3	1	2	3	4	5	6	7	8
1	2	4	6	8	2	4	6	8
2	3	4	7	8	3	4	7	8
3	4	8	4	8	4	8	4	8
4	5	6	7	8	5	6	7	8
5	6	8	6	8	6	8	6	8
6	7	8	7	8	7	8	7	8
7	8	8	8	8	8	8	8	8
8	1	2	3	4	5	6	7	8

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1	1

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1	2	2
2	1	2

A_2	1	2	3	4
1	2	4	2	4
2	3	4	3	4
3	4	4	4	4
4	1	2	3	4

A_3	1	2	3	4	5	6	7	8
1	2	4	6	8	2	4	6	8
2	3	4	7	8	3	4	7	8
3	4	8	4	8	4	8	4	8
4	5	6	7	8	5	6	7	8
5	6	8	6	8	6	8	6	8
6	7	8	7	8	7	8	7	8
7	8	8	8	8	8	8	8	8
8	1	2	3	4	5	6	7	8

$\pi_3(1) = 4$
$\pi_3(2) = 4$
$\pi_3(3) = 2$
$\pi_3(4) = 4$
$\pi_3(5) = 2$
$\pi_3(6) = 2$
$\pi_3(7) = 1$
$\pi_3(8) = 8$

A_0	1
1	1

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1	2	2
2	1	2

A_2	1	2	3	4
1	2	4	2	4
2	3	4	3	4
3	4	4	4	4
4	1	2	3	4

A_3	1	2	3	4	5	6	7	8
1	2	4	6	8	2	4	6	8
2	3	4	7	8	3	4	7	8
3	4	8	4	8	4	8	4	8
4	5	6	7	8	5	6	7	8
5	6	8	6	8	6	8	6	8
6	7	8	7	8	7	8	7	8
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$\pi_3(5) = 2$

$\pi_3(6) = 2$

$\pi_3(7) = 1$

$\pi_3(8) = 8$

A_4	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	2	12	14	16	2	12	14	16	2	12	14	16	2	12	14	16
2	3	12	15	16	3	12	15	16	3	12	15	16	3	12	15	16
3	4	8	12	16	4	8	12	16	4	8	12	16	4	8	12	16
4	5	6	7	8	13	14	15	16	5	6	7	8	13	14	15	16
5	6	8	14	16	6	8	14	16	6	8	14	16	6	8	14	16
6	7	8	15	16	7	8	15	16	7	8	15	16	7	8	15	16
7	8	16	8	16	8	16	8	16	8	16	8	16	8	16	8	16
8	9	10	11	12	13	14	15	16	9	10	11	12	13	14	15	16
9	10	12	14	16	10	12	14	16	10	12	14	16	10	12	14	16
10	11	12	15	16	11	12	15	16	11	12	15	16	11	12	15	16
11	12	16	12	16	12	16	12	16	12	16	12	16	12	16	12	16
12	13	14	15	16	13	14	15	16	13	14	15	16	13	14	15	16
13	14	16	14	16	14	16	14	16	14	16	14	16	14	16	14	16
14	15	16	15	16	15	16	15	16	15	16	15	16	15	16	15	16
15	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16
16	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16

A_4	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	2	12	14	16	2	12	14	16	2	12	14	16	2	12	14	16
2	3	12	15	16	3	12	15	16	3	12	15	16	3	12	15	16
3	4	8	12	16	4	8	12	16	4	8	12	16	4	8	12	16
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16	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16

A_4	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
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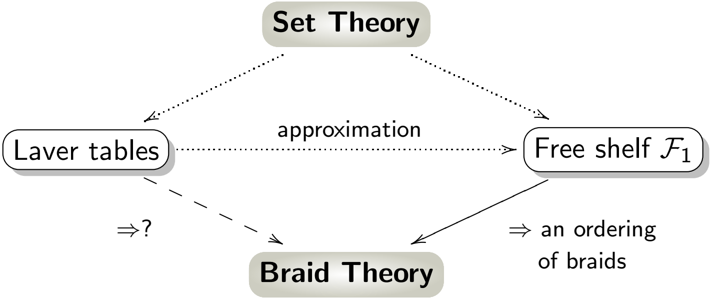
15	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16
16	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16

Rich combinatorics.

Part 2

Dreams: braid and knot invariants based on Laver tables

Laver tables in Topology

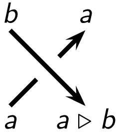


Richard Laver

Patrick Dehornoy

Shelf colorings

Colorings by $(S, \triangleright)$:



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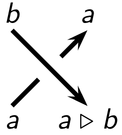

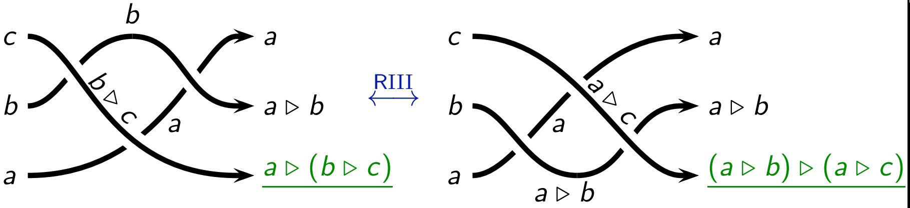

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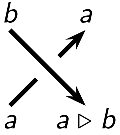

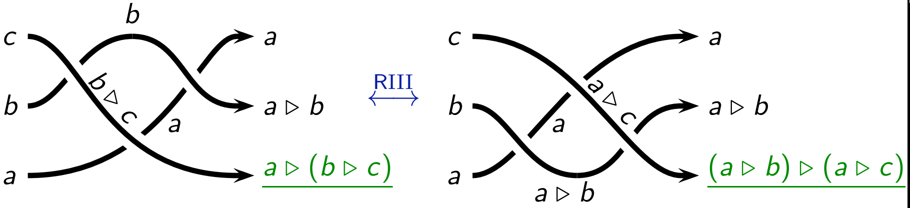

RIII $\leftrightarrow$ $a \triangleright (b \triangleright c) = (a \triangleright b) \triangleright (a \triangleright c)$ (SD)

positive braid invariants $\overset{\text{colorings}}{\leftsquigarrow}$ shelf

Shelf colorings

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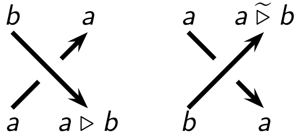

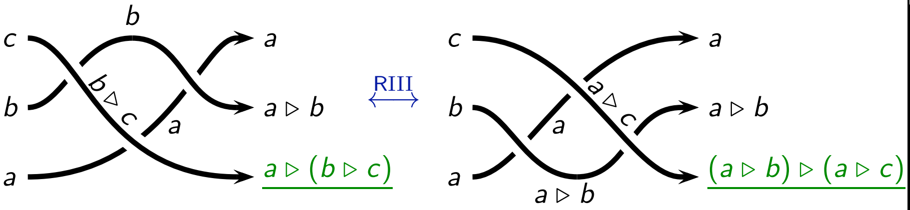

RIII	$\leftrightarrow$	$a \triangleright (b \triangleright c) = (a \triangleright b) \triangleright (a \triangleright c)$	(SD)
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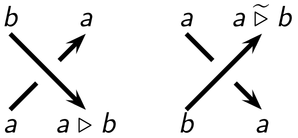

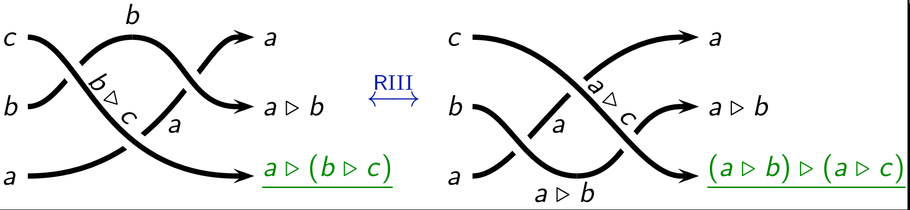

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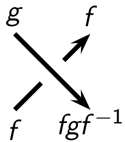
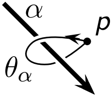
} **Rack**

braid invariants $\overset{\text{colorings}}{\leftsquigarrow}$ rack

Shelf colorings



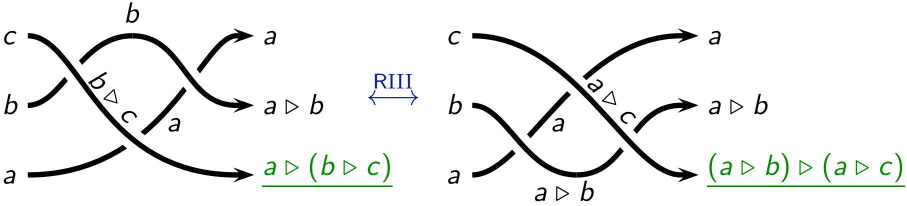
Wirtinger presentation:



colorings by G

$\updownarrow$

$Rep(\pi_1((\mathbb{R}^2 \times [0,1])\backslash\beta), G)$



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} **Rack**

braid invariants $\overset{\text{colorings}}{\rightsquigarrow}$ rack

Example: Group $G \rightsquigarrow$ a rack
$(G\,,\ f \triangleright g = fgf^{-1}\,,\ f \mathbin{\widetilde{\triangleright}} g = f^{-1}gf)$.

$\mathcal{F}_1$-colorings for arbitrary braids?

$$\text{positive braid invariants} \overset{\text{colorings}}{\leftsquigarrow} \mathcal{F}_1$$

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Question: What about arbitrary braid invariants?

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Problem: $\mathcal{F}_1$ is a shelf, but not a rack.
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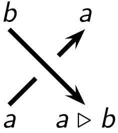
$$\sigma : S \times S \longrightarrow S \times S,$$
$$(a, b) \longmapsto (a \triangleright b, a).$$



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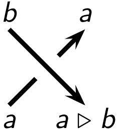

For $\mathcal{F}_1$, the map σ is only injective $\Longrightarrow$ partially invertible.

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Normal form for braids: $\boxed{\beta = \beta_-\beta_+}$, β_- negative, β_+ positive.

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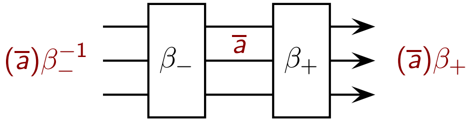

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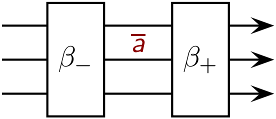
$(\overline{a})\beta_-^{-1}$  $(\overline{a})\beta_+$

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- $\forall$ k-braids β, β', $\exists$ a common propagable color vector $\overline{a}$.

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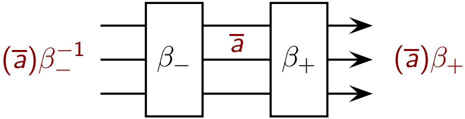

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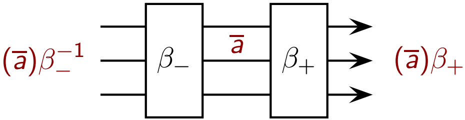

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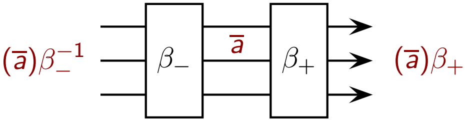

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$\mathcal{F}_1$-colorings for arbitrary braids?

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Remark:
- Alternative constructions of $<$.
- Applications: ✓ efficient algorithms for distinguishing braids,
  ✓ geometry of closed braids, etc.

A_n-colorings for arbitrary braids?

positive braid invariants $\overset{\text{colorings}}{\leftsquigarrow}$ Laver tables

Question: What about arbitrary braid invariants?

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Problem: The color propagation map σ is not even injective:

$$\sigma(2^n - 1, b) = ((2^n - 1) \triangleright b, 2^n - 1) = (2^n, 2^n - 1).$$

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❁ A_n are finite.

Part 3

Reality: 2- and 3-cocycles for Laver tables

Shelf colorings revisited

Aim: Add flexibility to coloring invariants.

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Method: enrich colorings with weights

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Rack cohomology (Fenn-Rourke-Sanderson, '95)

Shelf $(S, \triangleright) \rightsquigarrow$ complex $(\mathrm{Hom}(S^{\times k}, \mathbb{Z}), d_{\mathrm{R}}^k) \rightsquigarrow H_{\mathrm{R}}^k(S)$

$$(d_{\mathrm{R}}^k f)(a_1, \ldots, a_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i-1} (f(a_1, \ldots, a_{i-1}, a_i \triangleright a_{i+1}, \ldots, a_i \triangleright a_{k+1}) - f(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{k+1})).$$

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2-cocycles: maps $\phi : S \times S \to \mathbb{Z}$ satisfying

$$\boxed{\phi(a, c) + \phi(a \triangleright b, a \triangleright c) = \phi(b, c) + \phi(a, b \triangleright c)}$$

Cocycle invariants

Fix a 2-cocycle $\phi : S \times S \to \mathbb{Z}$:

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ϕ-weight (Carter-Jelsovsky-Kamada-Langford-Saito, '99):

$$\boxed{S\text{-colored diagram} \quad \longmapsto \quad \sum \phi(a, b)}$$

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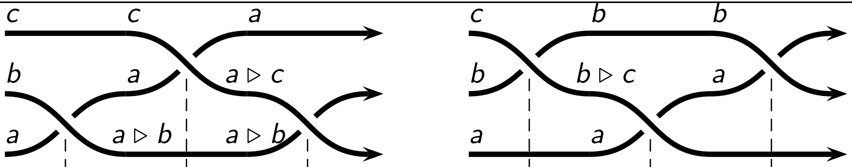

$\phi(a,b) + \phi(a,c) + \phi(a \triangleright b, a \triangleright c)$ $\qquad$ $\phi(b,c) + \phi(a, b \triangleright c) + \phi(a,b)$

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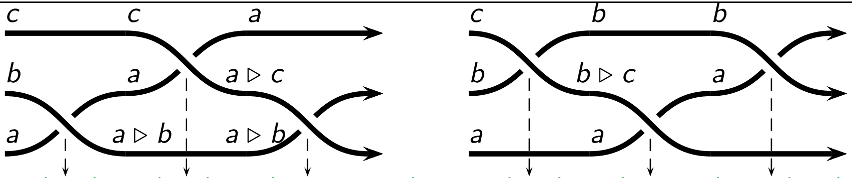

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positive braid invariants $\overset{\text{colorings \&}}{\underset{\text{weights}}{\leftsquigarrow}}$ shelf & 2-cocycle

Shadow cocycle invariants

Fix a 3-cocycle $\psi : S \times S \times S \to \mathbb{Z}$:

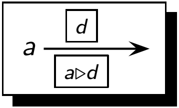
$$\boxed{\begin{array}{l}\psi(a, b, c \triangleright d) + \psi(a, c, d) + \psi(a \triangleright b, a \triangleright c, a \triangleright d) = \\ \quad \psi(b, c, d) + \psi(a, b \triangleright c, b \triangleright d) + \psi(a, b, d)\end{array}}$$

Shadow cocycle invariants

Fix a 3-cocycle $\psi : S \times S \times S \to \mathbb{Z}$:

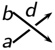
$$\boxed{\begin{array}{l}\psi(a,b,c \triangleright d)+\psi(a,c,d)+\psi(a\triangleright b, a\triangleright c, a\triangleright d)= \\ \quad \psi(b,c,d)+\psi(a,b\triangleright c, b\triangleright d)+\psi(a,b,d)\end{array}}$$

Shadow colorings:



ψ-weight:

S-colored diagram $\longmapsto \sum \psi(a,b,d)$



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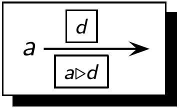

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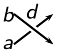

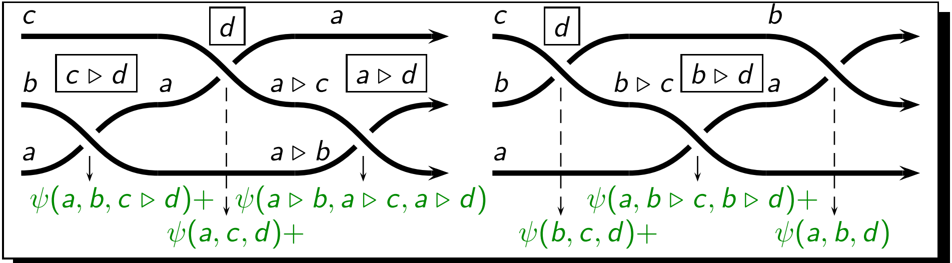

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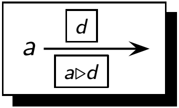

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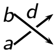

positive braid invariants $\overset{\text{colorings \&}}{\underset{\text{weights}}{\leftsquigarrow}}$ shelf & 3-cocycle

2- and 3-cocycles for Laver tables

Theorem (Dehornoy-L., '14)

1. $\boxed{Z^2_{\mathrm{R}}(A_n) \simeq \mathbb{Z}^{2^n}}$ basis: $\phi_{const}(a,b) = 1$ and coboundaries

$$\phi_{q,n}(a,b) = \begin{cases} 1 & \text{if } q \in Col(b),\ b \notin Col(a \triangleright b), \\ 0 & \text{otherwise.} \end{cases} \qquad 1 \leqslant q < 2^n$$

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2. $\boxed{Z^3_{\mathrm{R}}(A_n) \simeq \mathbb{Z}^{2^{2n} - 2^n + 1}}$ basis: $\psi_{const}(a,b,c) = 1$ and explicit $\{0, \pm 1\}$-valued coboundaries.

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Theorem (L., '14)

1. $\boxed{Z^k_{\mathrm{R}}(A_n) \simeq \mathbb{Z}^{P_k(2^n)}}$, $P_k(x) = \dfrac{x^k + x^{\alpha(k)}}{x+1}$, $\alpha(k) = \begin{cases} 1 & \text{if } k \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$
2. $\boxed{H^k_{\mathrm{R}}(A_n) \simeq \mathbb{Z}}$ for all k.

2- and 3-cocycles for Laver tables

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Remark: 2-cocycles capture the combinatorics of the A_n.

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Periods via cocycles

$$\pi_n(p) = \min \left\{ q \,\middle|\, \phi_{2^{n-1},n}(p,q) = 1 \right\}, \quad p < 2^n.$$

$\phi_{1,3}$	1
1	1
2	1
3	1
4	1
5	1
6	1
7	1
8	·

$\phi_{2,3}$	1	2	3	4	5	6	7	8
1	·	1	·	·	·	·	·	·
2	1	1	·	·	1	·	·	·
3	1	1	·	·	1	·	·	·
4	·	1	·	·	·	·	·	·
5	1	1	·	·	1	·	·	·
6	1	1	·	·	1	·	·	·
7	1	1	·	·	1	·	·	·
8	·	·	·	·	·	·	·	·

$\phi_{3,3}$	1	2	3	4	5	6	7	8
1	1	·	1	·	1	·	·	·
2	·	·	1	·	·	·	·	·
3	1	·	1	·	1	·	·	·
4	·	·	1	·	·	·	·	·
5	1	·	1	·	1	·	·	·
6	1	·	1	·	1	·	·	·
7	1	·	1	·	1	·	·	·
8	·	·	·	·	·	·	·	·

$\phi_{4,3}$	1	2	3	4	5	6	7	8
1	·	·	·	1	·	·	·	·
2	·	·	·	1	·	·	·	·
3	·	1	·	1	·	1	·	·
4	·	·	·	1	·	·	·	·
5	·	1	·	1	·	1	·	·
6	·	1	·	1	·	1	·	·
7	1	1	1	1	1	1	1	·
8	·	·	·	·	·	·	·	·

$\phi_{5,3}$	1	2	3	4	5	6	7	8
1	1	·	·	·	1	·	·	·
2	1	·	·	·	1	·	·	·
3	1	·	·	·	1	·	·	·
4	·	·	·	·	·	·	·	·
5	1	·	·	·	1	·	·	·
6	1	·	·	·	1	·	·	·
7	1	·	·	·	1	·	·	·
8	·	·	·	·	·	·	·	·

$\phi_{6,3}$	1	2	3	4	5	6	7	8
1	·	1	·	·	·	1	·	·
2	·	1	·	·	·	1	·	·
3	1	1	1	·	1	1	1	·
4	·	·	·	·	·	·	·	·
5	·	1	·	·	·	1	·	·
6	·	1	·	·	·	1	·	·
7	1	1	1	·	1	1	1	·
8	·	·	·	·	·	·	·	·

$\phi_{7,3}$	1	2	3	4	5	6	7	8
1	1	·	1	·	1	·	1	·
2	·	·	·	·	·	·	·	·
3	1	·	1	·	1	·	1	·
4	·	·	·	·	·	·	·	·
5	1	·	1	·	1	·	1	·
6	·	·	·	·	·	·	·	·
7	1	·	1	·	1	·	1	·
8	·	·	·	·	·	·	·	·

$\phi_{1,3}$	1
1	1
2	1
3	1
4	1
5	1
6	1
7	1
8	·

$\phi_{2,3}$	1	2	3	4	5	6	7	8
1	·	1	·	·	·	·	·	·
2	1	1	·	·	1	·	·	·
3	1	1	·	·	1	·	·	·
4	·	1	·	·	·	·	·	·
5	1	1	·	·	1	·	·	·
6	1	1	·	·	1	·	·	·
7	1	1	·	·	1	·	·	·
8	·	·	·	·	·	·	·	·

$\phi_{3,3}$	1	2	3	4	5	6	7	8
1	1	·	1	·	1	·	·	·
2	·	·	1	·	·	·	·	·
3	1	·	1	·	1	·	·	·
4	·	·	1	·	·	·	·	·
5	1	·	1	·	1	·	·	·
6	1	·	1	·	1	·	·	·
7	1	·	1	·	1	·	·	·
8	·	·	·	·	·	·	·	·

$\phi_{4,3}$	1	2	3	4	5	6	7	8
1	·	·	·	1	·	·	·	·
2	·	·	·	1	·	·	·	·
3	·	1	·	1	·	1	·	·
4	·	·	·	1	·	·	·	·
5	·	1	·	1	·	1	·	·
6	·	1	·	1	·	1	·	·
7	1	1	1	1	1	1	1	·
8	·	·	·	·	·	·	·	·

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1	1	·	·	·	1	·	·	·
2	1	·	·	·	1	·	·	·
3	1	·	·	·	1	·	·	·
4	·	·	·	·	·	·	·	·
5	1	·	·	·	1	·	·	·
6	1	·	·	·	1	·	·	·
7	1	·	·	·	1	·	·	·
8	·	·	·	·	·	·	·	·

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1	·	1	·	·	·	1	·	·
2	·	1	·	·	·	1	·	·
3	1	1	1	·	1	1	1	·
4	·	·	·	·	·	·	·	·
5	·	1	·	·	·	1	·	·
6	·	1	·	·	·	1	·	·
7	1	1	1	·	1	1	1	·
8	·	·	·	·	·	·	·	·

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1	1	·	1	·	1	·	1	·
2	·	·	·	·	·	·	·	·
3	1	·	1	·	1	·	1	·
4	·	·	·	·	·	·	·	·
5	1	·	1	·	1	·	1	·
6	·	·	·	·	·	·	·	·
7	1	·	1	·	1	·	1	·
8	·	·	·	·	·	·	·	·

Main theorem: sketch of proof

Theorem (Dehornoy-L., 14)

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$$\phi(2^n-1, 2^n) + \phi(2^n, 2^n) = \phi(b, 2^n) + \phi(2^n-1, 2^n)$$

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Row $2^n - 1$: $\widetilde{\phi}_{q,n}(2^n - 1, b) = \delta_{b,q}$.

$$\begin{array}{cccccc} 0 & 0 & 0 & 1 & 0 & 0 \\ & & & q & & \end{array}$$

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$$\boxed{\begin{matrix} 0 & 0 & 0 & 1 & 0 & 0 \end{matrix}}$$
$$\qquad\qquad\qquad q$$

$\Longrightarrow \left\{ \widetilde{\phi}_{q,n} \,\middle|\, 1 \leqslant q \leqslant 2^n \right\}$ is a basis of $Z^2_{\mathrm{R}}(A_n)$.

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Step 2. 2-cocycle constant on row $2^n - 1$ $\Longrightarrow$ constant.
Step 3. 2-coboundaries $\widetilde{\phi}_{q,n} = -d^2_{\mathrm{R}}(\delta_{q,\bullet}) : (a,b) \mapsto \delta_{b,q} - \delta_{a \triangleright b, q}$, $1 \leqslant q < 2^n$. Put $\widetilde{\phi}_{2^n,n} = \phi_{const} - \sum \widetilde{\phi}_{q,n}$.

Row $2^n - 1$: $\widetilde{\phi}_{q,n}(2^n - 1, b) = \delta_{b,q}$.

$$\begin{array}{cccccc} 0 & 0 & 0 & 1 & 0 & 0 \\ & & & q & & \end{array}$$

$\Longrightarrow \left\{ \widetilde{\phi}_{q,n} \,\middle|\, 1 \leqslant q \leqslant 2^n \right\}$ is a basis of $Z^2_{\mathrm{R}}(A_n)$.
Step 4. A change of basis.

$\widetilde{\phi}_{1,3}$	1	2	3	4	5	6	7	8
1	1	·	·	·	·	·	·	·
2	1	·	·	·	·	·	·	·
3	1	·	·	·	·	·	·	·
4	1	·	·	·	·	·	·	·
5	1	·	·	·	·	·	·	·
6	1	·	·	·	·	·	·	·
7	1	·	·	·	·	·	·	·
8	·	·	·	·	·	·	·	·

$\widetilde{\phi}_{4,3}$	1	2	3	4	5	6	7	8
1	·	−1	·	1	·	−1	·	·
2	·	−1	·	1	·	−1	·	·
3	−1	·	−1	1	−1	·	−1	·
4	·	·	·	1	·	·	·	·
5	·	·	·	1	·	·	·	·
6	·	·	·	1	·	·	·	·
7	·	·	·	1	·	·	·	·
8	·	·	·	·	·	·	·	·

$\widetilde{\phi}_{7,3}$	1	2	3	4	5	6	7	8
1	·	·	·	·	·	·	1	·
2	·	·	−1	·	·	·	·	·
3	·	·	·	·	·	·	1	·
4	·	·	−1	·	·	·	·	·
5	·	·	·	·	·	·	1	·
6	−1	·	−1	·	−1	·	·	·
7	·	·	·	·	·	·	1	·
8	·	·	·	·	·	·	·	·

Part 4

Bonus: right division ordering for Laver tables

Right division for Laver tables

Right division relation:

$$\boxed{a \,|_r\, b \quad \Longleftrightarrow \quad b = c \triangleright a \text{ for some } c}$$

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1. $|_r$ is a partial ordering for A_n.

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Properties:

- Minimal element: 1, maximal element: 2^n.

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- Linear ordering for $n \leqslant 2$, not linear for $n \geqslant 3$.

Right division for Laver tables

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$$\boxed{a \mid_r b \quad \iff \quad b = c \triangleright a \text{ for some } c}$$

Theorem (Dehornoy-L., 14)

1. $\mid_r$ is a partial ordering for A_n.
2. $a \mid_r b \quad \iff \quad Col(a) \supseteq Col(b)$.
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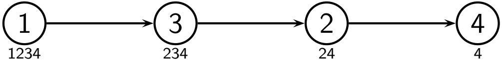
Properties:

- Minimal element: 1, maximal element: 2^n.
- Linear ordering for $n \leqslant 2$, not linear for $n \geqslant 3$.
- Lattice ordering for $n \leqslant 4$, not lattice for $n \geqslant 5$.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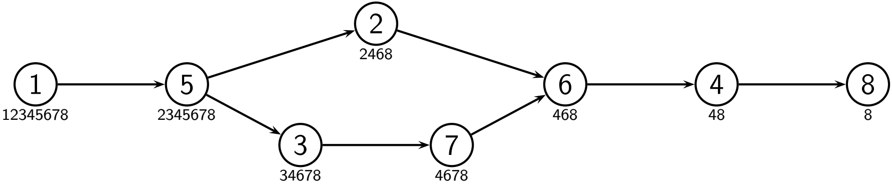
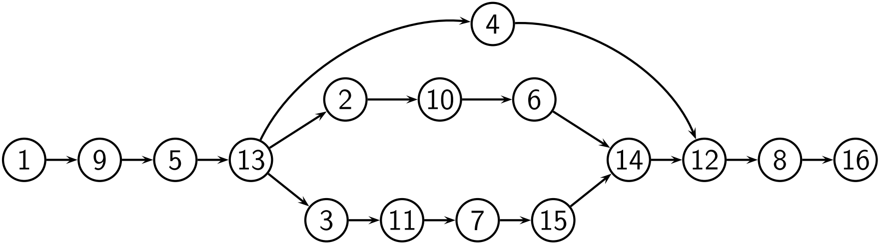
3
234
2
24
4
4


1
12345678
5
2345678
2
2468
3
34678
7
4678
6
468
4
48
8
8


1
9
5
13
4
2
10
6
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11
7
15
14
12
8
16

Main theorem 2: sketch of proof

Theorem (Dehornoy-L., 14)

$|_r$ is a partial ordering for A_n.

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Theorem (Dehornoy-L., 14)

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Step 1. Operation $p \circ q = p \triangleright (q+1) - 1$ satisfies

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$$a \,|_r\, b \,|_r\, c \quad \Rightarrow \quad b = d \triangleright a, \; c = e \triangleright b$$

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Step 7. Anti-symmetry of $|_r$.

A good basis for 2-cocycles

Theorem (Dehornoy-L., 14)

$\boxed{Z^2_{\mathrm{R}}(A_n) \simeq \mathbb{Z}^{2^n}}$ basis: $\phi_{const}(a, b) = 1$ and coboundaries

$$\phi_{q,n}(a, b) = \begin{cases} 1 & \text{if } q \in Col(b),\ b \notin Col(a \triangleright b), \\ 0 & \text{otherwise.} \end{cases} \qquad 1 \leqslant q < 2^n$$

We saw: ϕ_{const} and 2-coboundaries
$\widetilde{\phi}_{q,n} = -d^2_{\mathrm{R}}(\delta_{q,\bullet}) : (a, b) \mapsto \delta_{b,q} - \delta_{a \triangleright b, q} \in \{0, \pm 1\}$, $1 \leqslant q < 2^n$,
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Change of basis: $\boxed{\phi_{q,n} = \sum_{s|_r q} \widetilde{\phi}_{s,n}}$

Digression: Laver tables and branched braids

Theorem (Laver, Drápal, 95)

Operation $\boxed{p \circ q = p \triangleright (q+1) - 1}$ satisfies

$$(a \circ b) \triangleright c = a \triangleright (b \triangleright c), \qquad (a \circ b) \circ c = a \circ (b \circ c),$$
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$A_3, \circ$	1	2	3	4	5	6	7	8
1	3	5	7	1	3	5	7	1
2	3	6	7	2	3	6	7	2
3	7	3	7	3	7	3	7	3
4	5	6	7	4	5	6	7	4
5	7	5	7	5	7	5	7	5
6	7	6	7	6	7	6	7	6
7	7	7	7	7	7	7	7	7
8	1	2	3	4	5	6	7	8

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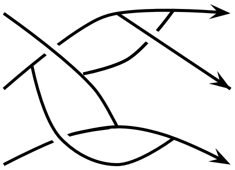
Application: colorings of positive **branched braids**.



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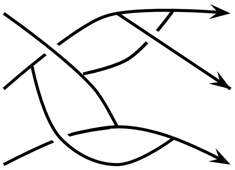

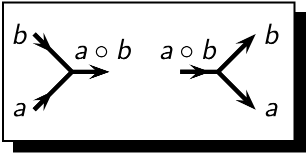

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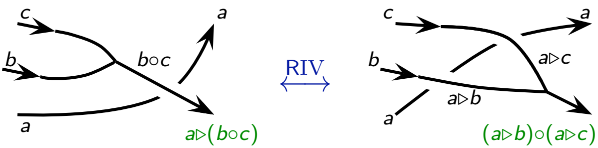

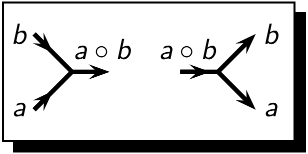

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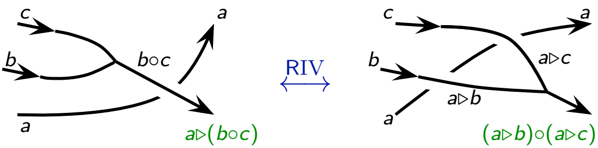

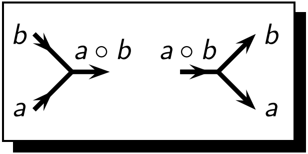

positive branched braid invariants $\overset{\text{colorings}}{\leftsquigarrow}$ A_n

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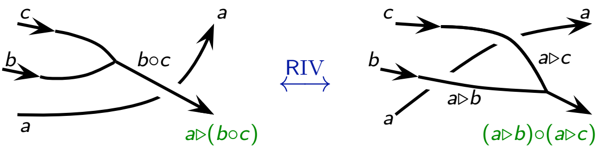

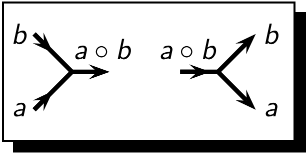

positive branched braid invariants $\overset{\text{colorings}}{\leftsquigarrow}$ A_n

⚠ Does not work for $\mathcal{F}_1$!

Division relations for shelves

	$a \mid_r b$ if $b = c \triangleright a$	$a \mid_l b$ if $b = a \triangleright c$
A_n	is a partial ordering $\rightsquigarrow$ a good basis for 2-cocycles	
$\mathcal{F}_1$		induces a total ordering $\rightsquigarrow$ an ordering of braids

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A_n	is a partial ordering $\rightsquigarrow$ a good basis for 2-cocycles	induces a trivial relation
$\mathcal{F}_1$		induces a total ordering $\rightsquigarrow$ an ordering of braids

Division relations for shelves

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Depth function: $d : \mathcal{F}_1 \to \mathbb{N}$, $d(\gamma) = 1$, $d(c \triangleright a) = d(a) + 1$.

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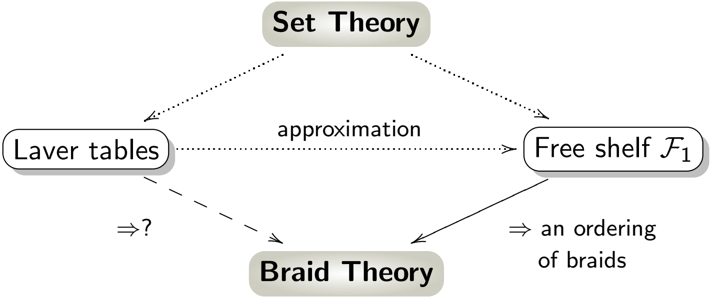
To be continued...



Richard Laver

Patrick Dehornoy