How braids can help to compute Hochschild cohomology

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Cohomological Methods in Geometry

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\[(ab)c = a(bc)\]

\[z^{-1}(y^{-1}xy)z = (z^{-1}y^{-1}z)(z^{-1}xz)(z^{-1}yz)\]
Yang–Baxter equation: classics

**Data**: vector space $V$, $\sigma: V^\otimes 2 \to V^\otimes 2$.

**Yang–Baxter equation (YBE)**

\[ \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 : V^\otimes 3 \to V^\otimes 3 \]

\[ \sigma_1 = \sigma \otimes \text{Id}_V, \sigma_2 = \text{Id}_V \otimes \sigma \]

**Avatars in:**

→ statistical mechanics;
→ quantum field theory;
→ algebra;
→ low-dimensional topology:

\[ \sigma \leftrightarrow \text{YBE} \leftrightarrow = \text{Reidemeister III move} \]
First deviation: braided sets

**Data**: set $S$, $\sigma: S^2 \to S^2$.

**Set-theoretic YBE** (Drinfel'd 1990)

\[
\sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2: S^3 \to S^3
\]

\[
\sigma_1 = \sigma \times \text{Id}_S, \quad \sigma_2 = \text{Id}_S \times \sigma
\]

Solutions are called **braided sets**.

Solutions linearise deform general solutions

**Examples**:

✓ $\sigma(x, y) = (x, y)$;

✓ $\sigma(x, y) = (y, x) \leadsto \text{R-matrices}$;

✓ **Lie algebra** $(V, [\,])$, central element $1 \in V$,

\[
\sigma(x \otimes y) = y \otimes x + \hbar 1 \otimes [x, y].
\]

YBE for $\sigma \iff$ Jacobi identity for $[\,]$
3 Self-distributivity

✓ set $S$, binary operation $\triangleleft$, $\sigma(x, y) = (y, x \triangleleft y)$

YBE for $\sigma \iff$ self-distributivity for $\triangleleft$

Self-distributivity: $(x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z)$

Examples:

→ group $S$ with $x \triangleleft y = y^{-1}xy$:

$$z^{-1}(y^{-1}xy)z = (z^{-1}y^{-1}z)(z^{-1}xz)(z^{-1}yz)$$

→ abelian group $S$, $t: S \to S$, $a \triangleleft b = ta + (1 - t)b$.

Applications:

→ invariants of knots and knotted surfaces;

→ a total order on braid groups;

→ Hopf algebra classification.
4. Even more exotic examples

✓ monoid \((S, *, 1)\), \(\sigma(x, y) = (1, x * y)\);

YBE for \(\sigma \iff\) associativity for \(*\)

✓ lattice \((S, \wedge, \vee)\), \(\sigma(x, y) = (x \wedge y, x \vee y)\).

All these braidings are idempotent: \(\sigma\sigma = \sigma\).
Universal enveloping monoids:

\[ \text{Mon}(S, \sigma) = \langle S \mid xy = y'x' \text{ whenever } \sigma(x, y) = (y', x') \rangle \]

U. e. groups and algebras are defined similarly.

Theorem: \((S, \sigma)\) a “nice” finite braided set, \(\sigma^2 = \text{Id} \implies \)

- \(\text{Mon}(S, \sigma)\) is of I-type, cancellative, Ore;
- \(\text{Grp}(S, \sigma)\) is solvable, Garside;
- \(\kappa \text{Mon}(S, \sigma)\) is Koszul, noetherian, Cohen–Macaulay, Artin–Schelter regular

(\text{Manin, Gateva-Ivanova & Van den Bergh, Etingof– Schedler–Soloviev, Jespers–Okniński, Chouraqui 80’–…}).
Example: \[ S = \{ a, b \}, \quad aa \xrightarrow{\sigma} bb \]

\[ \text{Grp}(S, \sigma) = \langle a, b \mid a^2 = b^2 \rangle =: G. \]

Realisation by Euclidean transformations of \( \mathbb{R}^2 \):

\[ b = a'ba' \]
\[ \downarrow a = a'b \]
\[ a^2 = b^2 \]

\( \mathbb{R}^2 / G \cong \text{Klein bottle} \):
Universal enveloping monoids:

\[ \text{Mon}(S, \sigma) = \langle S \mid xy = y'x' \text{ whenever } \sigma(x, y) = (y', x') \rangle \]

Examples:

✓ monoid \((S, *, 1), \quad \sigma(x, y) = (1, x * y),\)

\[ S \simeq \text{Mon}(S, \sigma)/1 = 1_{\text{Mon}}; \]

✓ Lie algebra \((V, [], 1), \quad \sigma(x \otimes y) = y \otimes x + h1 \otimes [x, y],\)

\[ \text{UEA}(V, []) \simeq \mathbb{k} \text{Mon}(V, \sigma)/1 = 1_{\text{Mon}}. \]
Representations

\[
\text{Mon}(S, \sigma) = \langle S \mid xy = y'x' \text{ whenever } \sigma(x, y) = (y', x') \rangle
\]

Representations of \((S, \sigma)\) := representations of \(k\text{ Mon}(S, \sigma)\),
i.e. vector spaces \(M\) with \(M \times S \to M\) s.t.

\[
(m \cdot x) \cdot y = (m \cdot y') \cdot x'
\]

Examples:

→ trivial rep.: \(M = k\), \(m \cdot x = m\);
→ \(M = k\text{ Mon}(S, \sigma)\), \(m \cdot x = mx\);
→ usual reps for monoids, Lie algebras, self-distributive structures.
A cohomology theory for braided sets should:

1) Describe diagonal deformations

$$\sigma_q(x, y) = q^{\omega(x, y)} \sigma(x, y), \quad \omega: S \times S \to \mathbb{Z}:$$

$$\omega \text{ a 2-cocycle} \implies \sigma_q \text{ a YBE solution.}$$

2) Yield knot and knotted surface invariants:

$$(S, \sigma)$$-coloured diagram $$(D, C)$$ & $$\omega: S \times S \to \mathbb{Z}$$

$$\leadsto$$ Boltzmann weight

$$B_\omega(C) = \sum_{y'} \omega(x, y') - \sum_{x'} \omega(x', y).$$

$$\omega \text{ a 2-cocycle} \implies \text{a knot invariant given by}$$

$$\{ B_\omega(C) | C \text{ is a } (S, \sigma)\text{-colouring of } D \}.$$
A cohomology theory?

A cohomology theory for braided sets should:

3) **Unify** cohomology theories for
   - associative structures,
   - Lie algebras,
   - self-distributive structures etc.

+ explain parallels between them,
+ suggest theories for new structures.

4) **Compute** the cohomology of $\mathbb{k} \text{Mon}(S, \sigma)$. 
Braided cohomology

Data: braided set \((S, \sigma)\) & bimodule \(M\) over it.

Construction:
\[
C^n(S, \sigma; M) = \text{Maps}(S^\times n, M),
\]
\[
d^n = \sum_{i=1}^{n+1} (-1)^{i-1} (d^n_{l; i} - d^n_{r; i}) : C^n \to C^{n+1},
\]
\[
x'_i \cdot f(x'_1 \ldots x'_{i-1} x_{i+1} \ldots x_{n+1})
\]
\[
d^n_{l; i} f: x_i x'_1 \ldots x'_{i-1} x_{i+1} \ldots x_{n+1}
\]
\[
\sigma_1 \ldots \sigma_{i-1} \uparrow
\]
\[
x_1 \ldots x_{n+1}
\]

Theorem:
\[
\Rightarrow d^{n+1} d^n = 0;
\]
\[
H^* (S, \sigma; M) \text{ is the \textbf{braided cohomology} of } (S, \sigma) \text{ with coefs in } M;
\]
\[
\Rightarrow \text{ for "nice" } M, \text{ there is a \textbf{cup product} } \cup: H^n \otimes H^m \to H^{n+m};
\]
\[
\Rightarrow \text{ other good properties.}
\]
A good theory?

1) & 2) For $\omega \in C^2(S, \sigma; \mathbb{Z})$,
\[ d^2 \omega = 0 \implies \omega \text{ yields Boltzmann weights} \]
\[ \& \text{ diagonal deformations,} \]
\[ \omega = d^1 \theta \implies \omega \text{ yields trivial...} \]

3) Unifies classical cohomology theories.

Example: monoid $(S, *, 1)$, $\sigma(x, y) = (1, x * y)$,
\[ d_{l}^{n;i} f: \]
\[ \ldots x_{i-2} x_{i-1} x_i x_{i+1} \ldots \xrightarrow{\sigma_{i-1}} \]
\[ \ldots x_{i-2} 1 (x_{i-1} * x_i) x_{i+1} \ldots \xrightarrow{\sigma_{i-2}} \]
\[ \ldots 1 x_{i-2} (x_{i-1} * x_i) x_{i+1} \ldots \xrightarrow{\ldots} \]
\[ 1 x_1 \ldots x_{i-2} (x_{i-1} * x_i) x_{i+1} \ldots \xrightarrow{\ldots} \]
\[ f(\ldots x_{i-2} (x_{i-1} * x_i) x_{i+1} \ldots) . \]
A good theory?

4) Quantum symmetriser $QS$:

\[
\begin{align*}
\text{braided cohomology} & \quad \text{of } (S, \sigma) \text{ with coefs in } M \\
\text{cup product} & \quad \text{QS} \\
\text{smaller complexes} & \quad \text{Hochschild cohomology} \\
& \quad \text{of } \mathbb{k} \text{Mon}(S, \sigma) \text{ with coefs in } M \\
& \quad \text{cup product} \\
& \quad \text{tools}
\end{align*}
\]

$QS$ is an isomorphism when
\[
\sigma \sigma = \text{Id} \quad \text{and} \quad \text{Char } \mathbb{k} = 0 \quad (\text{Farinati & García-Galofre 2016});
\]
\[
\sigma \sigma = \sigma \quad (L. \ 2016).
\]

Applications:
\[
\Rightarrow \text{factorisable groups},
\]
\[
\Rightarrow \text{Young tableaux}.
\]

Open problem: How far is $QS$ from being an iso in general?