Applications of self-distributivity to Yang–Baxter operators and their cohomology

Victoria LEBED, Trinity College Dublin (Ireland)

Busan, June 2017
Coloring invariants for braids

Self-distributivity: 
\[(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)\]

Diagram colorings by \((S, \triangleleft)\) for positive braids:

\[
\begin{align*}
&\quad c \to (a \triangleleft b) \triangleleft c \\
&\quad b \to b \triangleleft c \\
&\quad a \to c
\end{align*}
\]

\[
\begin{align*}
&\quad c \to (a \triangleleft c) \triangleleft (b \triangleleft c) \\
&\quad b \to b \triangleleft c \\
&\quad a \to c
\end{align*}
\]

\[
\begin{align*}
\text{End}(S^n) &\leftarrow B_n^+ \\
\text{RIII} &\sim
\end{align*}
\]

\[
\begin{align*}
&\quad (a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)
\end{align*}
\]

\[
\begin{align*}
&\quad \overline{a} \to \beta \\
&\quad \beta \to (\overline{a})\beta
\end{align*}
\]
Coloring invariants for braids

Diagram colorings by \((S, \triangleleft)\) for braids:

\[
\begin{align*}
\text{RII} & \sim \text{RII} \\
\text{RIII} & \sim \\
\text{End}(S^n) & \leftarrow B_n^+ \\
\text{Aut}(S^n) & \leftarrow B_n \\
S & \twoheadrightarrow (S^n)^B_n \\
a & \mapsto (a, \ldots, a)
\end{align*}
\]

\[
\begin{align*}
\text{RIII} & \quad (a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c) \\
\forall b, \ a & \mapsto a \triangleleft b \text{ invertible} \\
a \triangleleft a & = a
\end{align*}
\]

shelf, rack, quandle
Coloring invariants for braids

<table>
<thead>
<tr>
<th>End($S^n$) $\leftarrow$ $B_n^+$</th>
<th>RIII</th>
<th>$(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aut($S^n$) $\leftarrow$ $B_n$</td>
<td>&amp; RII</td>
<td>$\forall b, a \mapsto a \triangleleft b$ invertible</td>
</tr>
<tr>
<td>$S$ $\mapsto$ $(S^n)^B_n$</td>
<td></td>
<td>$a \triangleleft a = a$</td>
</tr>
<tr>
<td>$a \mapsto (a, \ldots, a)$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Examples:

<table>
<thead>
<tr>
<th>$S$</th>
<th>$a \triangleleft b$</th>
<th>$(S, \triangleleft)$ is a</th>
<th>in braid theory</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{Z}[t^{\pm 1}]$ Mod</td>
<td>$ta + (1-t)b$</td>
<td>quandle</td>
<td>(red.) Burau: $B_n \rightarrow \text{GL}_n(\mathbb{Z}[t^{\pm 1}])$</td>
</tr>
</tbody>
</table>

$$\rho_B(\begin{array}{c}
\begin{array}{c}
\vdots \\

\vdots \\

\vdots \\

1 \\
\end{array}
\end{array}) = I_{i-1} \oplus \begin{pmatrix}
1 - t & 1 \\
t & 0
\end{pmatrix} \oplus I_{n-i-1}$$
## Coloring invariants for braids

<table>
<thead>
<tr>
<th>End($S^n$) $\leftarrow B^+_n$</th>
<th>RIII</th>
<th>$(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aut($S^n$) $\leftarrow B_n$</td>
<td>&amp; RII</td>
<td>$\forall b, a \mapsto a \triangleleft b$ invertible</td>
</tr>
<tr>
<td>$S \mapsto (S^n)^B_n$</td>
<td></td>
<td>$a \triangleleft a = a$</td>
</tr>
</tbody>
</table>

### Examples:

<table>
<thead>
<tr>
<th>$S$</th>
<th>$a \triangleleft b$</th>
<th>$(S, \triangleleft)$ is a</th>
<th>in braid theory</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{Z}[t^{\pm 1}]$ Mod</td>
<td>$ta + (1 - t)b$</td>
<td>quandle</td>
<td>(red.) Burau: $B_n \rightarrow \text{GL}_n(\mathbb{Z}[t^{\pm}])$</td>
</tr>
<tr>
<td>group</td>
<td>$b^{-1}ab$</td>
<td>quandle</td>
<td>Artin: $B_n \leftrightarrow \text{Aut}(F_n)$</td>
</tr>
<tr>
<td>twisted linear quandle</td>
<td></td>
<td></td>
<td>Lawrence–Krammer–Bigelow</td>
</tr>
<tr>
<td>$\mathbb{Z}$</td>
<td>$a + 1$</td>
<td>rack</td>
<td>$\text{lg}(w), \text{lk}_{i,j}$</td>
</tr>
<tr>
<td>free shelf</td>
<td></td>
<td></td>
<td>Dehornoy: order on $B_n$</td>
</tr>
<tr>
<td>Laver tables</td>
<td></td>
<td></td>
<td>???</td>
</tr>
</tbody>
</table>
Coloring counting invariants for knots

Diagram colorings by $(S, \prec)$ for knots:

\[
\begin{align*}
\text{pos. braids} & \quad \text{RIII} & \quad (a \prec b) \prec c = (a \prec c) \prec (b \prec c) \\
\text{braids} & \quad \& \text{RII} & \quad \forall b, \ a \mapsto a \prec b \text{ invertible} \\
\text{knots} \& \text{links} & \quad \& \text{RI} & \quad a \prec a = a
\end{align*}
\]
Theorem (Joyce & Matveev ’82):

✓ The number of colorings of a diagram $D$ of a knot $K$ by a quandle $(S, \lhd)$ yields a knot invariant.

✓ $\# \text{Col}_{S, \lhd}(D) = \# \text{Hom}_{\text{Quandle}}(Q(K), S) = \text{Tr}(\rho_S(\beta))$

- $Q(K) =$ fundamental quandle of $K$
  (a weak universal knot invariant);
- $\text{closure}(\beta) = K$;
- $\rho_S : B_n \to \text{Aut}(S^n)$ is the $S$-coloring invariant for braids.
Enhancing invariants: weights

Fenn–Rourke–Sanderson ’95 & Carter–Jelsovsky–Kamada–Langford–Saito ’03:

Shelf $S$, $\phi: S \times S \to \mathbb{Z}_n \leadsto \phi$-weights:

$$S\text{-colored diagram } D \mapsto \sum_{a,b} \pm \phi(a, b)$$

The multi-set of weights yields a braid invariant iff

$$\phi(a, b) + \phi(a \triangleleft b, c) + \phi(b, c) =$$

and a knot invariant if moreover $\phi(a, a) = 0$. 

\[ \phi(b, c) + \phi(a, c) + \phi(a \triangleleft c, b \triangleleft c) \]
Enhancing invariants: weights

These $\phi$-weights strengthen coloring invariants.

**Example:** $S = \{0, 1\}$, $a < b = a,$

$$\phi(0, 1) = 1 \text{ and } \phi(a, b) = 0 \text{ elsewhere.}$$

Conjecture (**Clark–Saito–...**): Finite quandle cocycle invariants distinguish all knots.

More generally, this approach works for knottings $K^{n-1} \hookrightarrow \mathbb{R}^{n+1}$.
4. Self-distributive cohomology

$$C^k_R(S, \mathbb{Z}_n) = \text{Map}(S \times^k, \mathbb{Z}_n),$$

$$(d^k_R f)(a_1, \ldots, a_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i-1} (f(a_1, \ldots, \hat{a_i}, \ldots, a_{k+1})$$

$$- f(a_1 \triangleleft a_i, \ldots, a_{i-1} \triangleleft a_i, a_{i+1}, \ldots, a_{k+1}))$$

$\leadsto$ Rack cohomology $H^k_R(S, \mathbb{Z}_n)$.

Applications:

1. (Higher) braid and knot invariants:
   
   $$d^2_R \phi = 0 \implies \phi \text{ refines (positive) braid coloring invariants},$$
   
   $$\phi = d^1_R \psi \implies \text{the refinement is trivial}.$$


3. Rack/quandle extensions, deformations etc.
Diagram colorings by \((S, \sigma)\):
\[
\begin{align*}
\sigma(a, b) &= (b_a, a^b) \\
Ex.: \sigma_\triangleleft(a, b) &= (b, a \triangleleft b)
\end{align*}
\]

RIII-compatibility \iff set-theoretic Yang-Baxter equation:
\[
\sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2: S^{\times 3} \to S^{\times 3}
\]
\[
\sigma_1 = \sigma \times \text{Id}_S, \quad \sigma_2 = \text{Id}_S \times \sigma
\]

Set-theoretic solutions linearize \iff deform linear solutions.

Example: \(\sigma(a, b) = (b, a)\) \(\leadsto\) \(\text{R-matrices}\).
Diagram colorings by \((S, \sigma)\):

\[
\begin{array}{c}
\sigma(a, b) = (b_a, a^b) \\
\sigma_\triangle(a, b) = (b, a \triangle b)
\end{array}
\]

RIII-compatibility \iff set-theoretic Yang-Baxter equation:

\[\sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 : S^3 \to S^3\]

Exotic example:

\[\sigma_{\text{Lie}}(a \otimes b) = b \otimes a + h_1 \otimes [a, b],\] where \([1, a] = [a, 1] = 0:\]

Very exotic example:

\[\sigma_{\text{Ass}}(a, b) = (a * b, 1),\] where \(1 * a = a:\]
Diagram colorings by \((S, \sigma)\):

\[
\begin{align*}
&b \quad \sigma(b) = (b, a^b) \\
&a \quad \sigma(a) = (b_a, a^b)
\end{align*}
\]

Ex.: \(\sigma_{\triangle}(a, b) = (b, a \triangle b)\)

<table>
<thead>
<tr>
<th>RIII</th>
<th>(\sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2)</th>
</tr>
</thead>
</table>
| & RII | \(\sigma\) invertible & \\
|      | \(\forall b, a \mapsto a^b\) and \(a \mapsto a_b\) invertible |
| & RI  | \(\exists a\) bijection \(t\) \\
|      | such that \(\sigma(t(a), a) = (t(a), a)\) |

**Result:** Coloring invariants of braids and knots.

**Bad news:** These invariants give nothing new!

**Unrelated question:** Describe free biracks and biquandles.
From biracks to racks

**Thm (Soloviev & Lu–Yan–Zhu ’00, L.–Vendramin ’17):**

✓ Birack $(S, \sigma)$ $\leadsto$ its structure rack $(S, \triangleleft_{\sigma})$:

\[
\begin{align*}
\sigma &\quad a \\
\sigma &\quad b \\
\sigma &\quad a \triangleleft_{\sigma} b
\end{align*}
\]

✓ This is a projection **Birack** $\rightarrow$ **Rack** along involutive biracks:

- $\triangleleft_{\sigma} \triangleleft = \triangleleft$;
- $\triangleleft_{\sigma}$ trivial $\iff \sigma^2 = \text{Id.}$

✓ The structure rack remembers a lot about the birack:

- $(S, \triangleleft_{\sigma})$ quandle $\iff (S, \sigma)$ biquandle;
- $\sigma$ and $\triangleleft_{\sigma}$ induce isomorphic $B_n$-actions on $S^n$ $\implies$ same braid and knot invariants.
From biracks to racks

Operation $\triangleleft_{\sigma}$ is self-distributive:

$$(a \triangleleft_{\sigma} b) \triangleleft_{\sigma} c = (a \triangleleft_{\sigma} c) \triangleleft_{\sigma} (b \triangleleft_{\sigma} c)$$
Guitar map

\[ J : S^{\times n} \xrightarrow{1:1} S^{\times n}, \]

\[ (x_n, \ldots, x_1) \mapsto (\ldots, (x_3)_{x_2} x_1, (x_2)_{x_1} x_1, x_1). \]
Guitar map

\[ J: S \times n \xrightarrow{1:1} S \times n, \]
\[ (x_n, \ldots, x_1) \mapsto (\ldots, (x_3)_{x_2 x_1}, (x_2)_{x_1}, x_1). \]

**Ex.** \( \sigma_{\text{Ass}}(a, b) = (ab, 1) \sim J(a, b, c) = (a, ab, abc). \)

**Ex.** \( \sigma_{\text{SD}}(a, b) = (b \triangleleft a, a) \sim J(a, b, c) = (a, b \triangleleft a, (c \triangleleft b) \triangleleft a). \)

**Ex.** \(\sigma^2 = 1d \sim \Omega \) from right-cyclic calculus.
**Guitar map**

\[ J: S \times n \xrightarrow{1:1} S \times n, \]
\[(x_n, \ldots, x_1) \mapsto (\ldots, (x_3)_{x_2}x_1, (x_2)x_1, x_1).\]

**Proposition:** \[ J\sigma_i = \sigma_i'J. \]

\[ \sigma: \begin{array}{c}
    b
    \end{array} \xrightarrow{\sigma} \begin{array}{c}
    a^b
    \end{array} \]
\[ \sigma': \begin{array}{c}
    b
    \end{array} \xleftarrow{\sigma} \begin{array}{c}
    a
    \end{array} \]

**Corollary:** Same \( B_n \)-actions and knot invariants.

\[ \text{⚠️} \ (S, \sigma) \nRightarrow (S, \sigma') \text{ as biracks!} \]
Proposition: $J\sigma_i = \sigma'_i J$.

Proof:

\[
\begin{align*}
J_4(\sigma_2(\bar{x})) & = J_4(\bar{x}) \\
J_3(\sigma_2(\bar{x})) & = J_2(\bar{x}) \triangleleft J_3(\bar{x}) \\
J_2(\sigma_2(\bar{x})) & = J_3(\bar{x}) \\
J_2(\bar{x}) & = J_3(\bar{x}) \\
J_1(\sigma_2(\bar{x})) & = J_1(\bar{x})
\end{align*}
\]
Braided cohomology

Carter–Elhamdadi–Saito ’04 & L. ’13:

\[ C^k_{Br}(S, \mathbb{Z}_n) = \text{Map}(S \times k, \mathbb{Z}_n), \]

\[ (d^k_{Br} f)(a_1, \ldots, a_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i-1} f(a_1, \ldots, a_{i-1}, (a_{i+1}, \ldots, a_{k+1})a_i) \]

\[ - f((a_1, \ldots, a_{i-1})a_i, a_{i+1}, \ldots, a_{k+1}) \]

\[ = \sum (-1)^{i-1} \left( f \right) \]

\[ \leadsto \text{Braided cohomology } H^k_{Br}(S, \mathbb{Z}_n). \]
Why I like braided cohomology

1. (Higher) braid and knot invariants:

\[ d^2_{\text{Br}} \phi = 0 \implies \phi \text{ refines (positive) braid coloring invariants,} \]
\[ \phi = d^1_{\text{Br}} \psi \implies \text{the refinement is trivial.} \]

Question: New invariants?

Answer: I don’t know!

2. \[ d^2_{\text{Br}} \phi = 0 \implies \text{diagonal deformations of } \sigma: \]
\[ \sigma_q(a, b) = q^{\phi(a, b)} \sigma(a, b). \]

(Freyd–Yetter ’89, Eisermann ’05)
Why I like braided cohomology

Unifies cohomology theories for

- self-distributive structures
  \[ \sigma_{SD}(a, b) = (b \triangleleft a, a) \]

- associative structures
  \[ \sigma_{Ass}(a, b) = (a \ast b, 1) \]

- Lie algebras
  \[ \sigma_{Lie}(a \otimes b) = b \otimes a + H1 \otimes [a, b] \]

+ explains parallels between them,

+ suggests theories for new structures.
Why I like braided cohomology

For certain $\sigma$, computes the group cohomology of

$$\text{Grp}(S, \sigma) = \langle S \mid ab = b_{a}a^{b}, \text{ where } \sigma(a, b) = (b_{a}, a^{b}) \rangle$$

Example: $\text{Grp}(S, \sigma_{\text{SD}}) = \langle S \mid a \ b = b \ (a \triangleleft b) \rangle = \text{As}(S, \triangleleft)$.

Applications: Cohomology of factorized groups & plactic monoids.

Rmk: $\text{Grp}(S, \sigma)$-modules are coefficients for braided cohomology (“walls”).

Rmk: Structure racks know a lot about structure groups.
Sideways maps: \( a \cdot b \mapsto a \tilde{} b \)

\[ a \quad \mapsto \quad b \]

**Fenn–Rourke–Sanderson ’93, Ceniceros–Elhamdadi–Green–Nelson ’14:**

\[ C^k_{\text{Bir}}(S, \mathbb{Z}_n) = \text{Map}(S \times^k, \mathbb{Z}_n), \]

\[
(d^k_{\text{Bir}} f)(a_1, \ldots, a_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i-1} (f(a_1, \ldots, \hat{a}_i, \ldots, a_{k+1})
- f(a_i \tilde{} a_1, \ldots, a_i \tilde{} a_{i-1}, a_i \cdot a_{i+1}, \ldots, a_i \cdot a_{k+1}))
\]

\( \leadsto \) Birack cohomology \( H^k_{\text{Bir}}(S, \mathbb{Z}_n). \)

**Normalized subcomplex** \( C^k_N \) for biquandles: \( f(\ldots, a_i, a_i, \ldots) = 0. \)

**Application:** Braid and knot invariants.
Guitar map counter-attacks

Thm (L.–Vendramin ’17):

✓ Braided and birack cohomologies are the same:

\[ J^* : (C^\bullet_{\text{Bir}}(S, \mathbb{Z}_n), d^\bullet_{\text{Bir}}) \cong (C^\bullet_{\text{Br}}(S, \mathbb{Z}_n), d^\bullet_{\text{Br}}). \]

✓ For biquandles, cohomology decomposes: \( C^\bullet_{\text{Bir}} \cong C^\bullet_N \oplus C^\bullet_D. \)

Question: Does \( C^\bullet_N \) determine \( C^\bullet_D \)?

Particular cases:

✓ a rack \((X, \triangleleft)\) and its dual \((X, \tilde{\triangleleft})\) have the same cohomology (folklore);
✓ cohomology decomposition for quandles (Litherland–Nelson ’03);
✓ two forms of group cohomology (folklore);
✓ new results for involutive biracks.

Proof: Use a graphical version of \( d^\bullet_{\text{Bir}} \) & play with diagrams!