I. Structural pre-braidings

- Algebraic structure + modules → case by case

II. Braided homology

- (pre-) braiding + braided module → canonical

(\text{bi-})complex → homology & other info

1. Unitary associative algebra + algebra module

2. (self-distributive) structure + rack-set

3. Lie/Leibniz algebras

4. Bialgebras

5. Hopf & Yetter-Drinfel'd (bi-)modules etc.

III. Some refinements
I. Structural pre-braidings.

We work in a strictly monoidal category $\mathcal{C}$ (often $\text{Vec}_k$, $\text{Mod}_R$, $\text{Mod}_k(R)$).

Pre-braided object: $(V^e, \varepsilon: V \otimes V \rightarrow V)$ + YBE.

Braided module over $(V, \varepsilon)$: $(M^e, \rho: M \otimes V \rightarrow M)$ + Braided character: braided module $(I, \varepsilon: I \rightarrow I)$.

Remark: left = right.

Categories: $\text{Br}(\mathcal{C})$, $\text{Mod}(\mathcal{C})$,

$\text{Br}^\ast(\mathcal{C})$, $\text{Mod}^\ast(\mathcal{C})$.

with a chosen $\ast$ "acts by identity" element $r: I \rightarrow V$.

Prop. 1: $\text{UAA}(\mathcal{C}) \xrightarrow{\text{fully faithful functor}} \text{Br}^\ast(\mathcal{C})$

$(V, \mu, \nu) \xrightarrow{f} (V, \nu_{\ast\ast} = \nu \otimes \mu, \nu)$

Prop. 2: $\text{Set} \xrightarrow{\text{fully faithful functor}} \text{Br}(\text{Set})$

$(S, \ast) \xrightarrow{f} (S, \ast \ast)$

- Associativity of $\mu \iff \text{YBE for } \nu_{\ast\ast}$
- $\text{Mod}^\ast_S \xrightarrow{\sim} \text{Mod}(\mathcal{C})$
- Highly non-invertible: $\nu_{\ast\ast} \neq \nu_{\ast\ast}$

- $\text{Sp for } \ast \iff \text{YBE for } \nu_{\ast\ast}$
- $\text{Mod}^\ast_S \xrightarrow{\sim} \text{Mod}(S, \mathcal{C})$
- $\text{Sp}$ for $\ast \iff \text{in invertibility}$

for $\mathcal{C}$.
II. Braided homology

1. Th. In a strictly mon. pre-additive cat. C, take:
   - a braided object $(V, \otimes)$
   - a right module $(M, p)$ over $(V, \otimes)$
   - a left $-(V, \otimes) \rightarrow (N, \otimes)$

Then $C_n = M \otimes V^n \otimes N$ can be endowed with a pre-bisimplicial structure

\[ p_{d_n; i} = \]

\[ d_{n; i} = \]

Remark: Reasoning in terms of strands, one avoids the tiresome index-chasing.

Proof:

\[ \text{braided module} \]

\[ \text{YBE:} \]

\[ \text{To take into account the (-1)^i sign, replace } \& \text{ with } \& \text{ i.e. count the crossings.} \]

\[ \text{We need local (hence simpler) properties rather than the global ones (i.e. of simplicial) structure.} \]

Cor: \((p_{d_n; i} = \sum_{i=1}^{n} (-1)^i \cdot d_{n; i})\), \((d_{n; i} = \sum_{i=1}^{n} (-1)^i \cdot d_{n; i})\) is a bi-differential on $C_n$.

Ex: 1. UAA: \(p_{d_n; i} = \frac{m}{n} \cdot v_{i-1} \cdot v_i \cdot v_{i+1} \ldots \)

2. S D:

\[ m, s_{13}, s_{23}, s_{12}, \ldots \]

\[ m, s_{12}, s_{13}, s_{23}, \ldots \]

\( V = RS \) or \( RS \)

\[ \Rightarrow p_d = \text{bar diff}^2 \text{ with coefficients on the left.} \]

\[ \Rightarrow p_d = 1\text{-term distributive diff}^3 \text{ with coeff}^3 \text{ on the left.} \]

\[ \varepsilon_d - d^2 = \text{rack diff}^3 \]

\[ \varepsilon_d - d^2 = \text{twisted rack diff}^3 \]

\[ \varepsilon: \alpha_1 \rightarrow 1 \text{ if } \alpha \in S \]

\( \Rightarrow \) braided character
Pre-braided coalgebra \((V, \varepsilon, \Delta)\) in \(C\):
- pre-br. object \((V, \varepsilon)\);
- co-associ. coalg. \((V, \Delta)\);
- compatibilities: \(\Delta^2 = \Delta\Delta = \Delta\).

Semi-braided coalgebra: only \(\Delta^2 = \Delta\).

\(\varepsilon\)-cocommutativity: \(\Delta^\varepsilon = \varepsilon\).

**Th. Bis.** If moreover \((V, \varepsilon, \Delta)\) is a pre-braided coalgebra, then
\(\left( \bigwedge_n, P^+_{n,i}, d_{n,i}^{n,i}, S_{n,i}^i \right) = \bigwedge_{i=1}^n N_{i=1}^N \) is a very weakly bisimplicial structure, becoming weakly simplicial if \(\Delta\) is \(\varepsilon\)-cocommutative.

If \((V, \varepsilon, \Delta)\) is only semi-braided, one should work with \((\bigwedge_n, P^+_{n,i}, S_{n,i}^i)\).

\(\text{Cpt. } \frac{\alpha}{\beta} \rightarrow \text{Im}(\beta_{n,i}^i) \) is a sub-bicomplex, called degenerate.

**Ex:**
- \(\text{UAA: } \Delta_4: a \mapsto 1 \otimes a\).
  - Lemma: \((V, \varepsilon, \Delta_3)\) is a pre-br. \(\varepsilon\)-cocomm coalgebra.
  \(P_n = \text{span} \{ x \otimes \omega_{i_1} \otimes \ldots \otimes \omega_{i_n} | 1 \leq i_1 \leq \ldots \leq i_n \leq n \}; \left( c_n, P_n, d_n, d_n^3 \right) / P_n \approx \text{Hochschild c-x}\)

- \(\text{So. } \Delta_5: a \mapsto a \otimes a\).
  - Lemma: \(\rightarrow \text{semi pre-braided}\)
    - \(\text{pre-braided } \Rightarrow a \otimes b = (a \otimes b) \otimes b + b \otimes a \in S\)
    - \(\varepsilon\)-cocomm. \(\Rightarrow S\) is a spindle: \(a \otimes a = a \otimes a \in S\)

\(P_n = \text{span} \{ (a_{i_1}, a_{i_2}, a_{i_3}, a_{i_4}, a_{i_5}) | 1 \leq i_1 \leq \ldots \leq i_5 \}; \left( c_n, P_n, d_n, d_n^5 \right) / P_n \approx \text{quandle c-x}\)


**Concatenation:**

\[
\begin{align*}
\text{Ex.: } & (a_1 \cdot b_1) \cdot (a_2 \cdot b_2) = a_1 b_1 \cdot a_2 b_2 \\
& \text{such that } a_1, a_2, b_1, b_2 \in \mathbb{R}^n.
\end{align*}
\]

**Arrow operation:**

\[
\begin{align*}
\text{Ex.: } & (a_1, b_1) \cdot (a_2, b_2) = (a_1 b_2, a_2 b_1) \\
& \text{such that } a_1, a_2, b_1, b_2 \in \mathbb{R}^n.
\end{align*}
\]

**Prop.:**

\[
\begin{align*}
& d^2 \circ T_y = e^2 = \text{Id} \quad \text{if } \quad x_0 = \frac{1}{2} \\
& d^2 \circ T_y = e^2 = \text{Id} \quad \text{if } \quad x_0 = \frac{1}{2} \\
& T_y = \text{Id} \quad \text{if } \quad x_0 = \frac{1}{2}
\end{align*}
\]

Ex.: $a+b$ has no zero divisors, e.g., $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{C}$.

Ex.: $a+b = a \cdot b$, $a, b \in \mathbb{R}$.

Ex.: $a \cdot b = a = b$.

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Some refinements

1. Pre-braided system: $V_i, -i, V_j, V_i \otimes V_j \rightarrow V_j \otimes V_i + \text{i} \leq j$.
   + YBE on $V_i \otimes V_j \otimes V_k$ for $i = j \leq k$.

   $\Rightarrow$ bialgebras, Hopf & YD (bi) modules etc.
   $\Rightarrow$ multi-braided tensor products of algebras

2. Multi-braided object $(V; \varepsilon_i, \varepsilon_r; \omega; V \otimes V \otimes \ldots)$ + mixed YBE:

   $\varepsilon_i = \varepsilon_r \quad \forall 1 \leq i, j \leq r$.

   Ex: a set $S, \varepsilon_i = \emptyset, \forall \geq \emptyset \in S$; all mixed YBE $\Rightarrow$ multi-distributivity.

Multi-braided module: $(M, \rho_i: M \otimes D \rightarrow M, 1 \leq i \leq r) + \rho_i$.

Cf. (*) for the case $M=I$.

Ex: 2 partial derivatives $\partial \times \partial$.

Th. multi:
- $(M \otimes \partial, \partial D, \partial D)$ is a differential multi-complex.
- pre-multisimplicial & weakly multisimplicial structure.

Ex: 2 multi-term distributive differential.