Rough notes for Maths 445.11[1]

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Since spatial rotations are Lorentz transformations, you might expect that there is a $\mathfrak{so}(3)$ subalgebra of $\mathfrak{so}(3,1)$. This is the case, define

$$l_1 = l_{23}, \qquad l_2 = l_{31}, \qquad l_3 = l_{12} \tag{1}$$

then

$$[l_i, l_j] = -\epsilon_{ijk} l_k. \tag{2}$$

This is the $\mathfrak{so}(3)$ algebra[2]. The rest of the algebra doesn't close on itself however:

$$[l_{01}, l_{02}] = g_{00} l_{12} = l_{12} = l_3 \tag{3}$$

and so on. Nonetheless, the $\mathfrak{so}(3,1)$ is the direct sum of two $\mathfrak{so}(3)$ provided we work with the complexified algebra.

Recall that a Lie algebra is a vector space with a closed skewsymmetric bilinear form. The Lie algebras we have been looking at have all been derived from groups, and when they are derived from groups, the vector space V is real. In other words it consists of vectors satisfying $\lambda u + \mu v \in V$ for u and v in V and λ and μ real. We have on occasion strayed outside the real algebra, the ladder operators $J_\pm$ aren't elements of the real algebra, they are elements of the corresponding *complexified* algebra, formed by taking the real algebra and allowing complex as well as real combinations of the elements.[3] However, the representations we eventually produced using the ladder operators are used to generate the real algebra. In the case we are looking at here, the Lorentz algebra, the relationship we will now discuss is only valid for the complexified algebras. The idea is that the complexified algebras are what are important in the quantum field theory context.

So, considering the complexified algebra, consider

$$l_i^\pm = \frac{1}{2}\left(l_i \pm i l_{0i}\right). \tag{4}$$

The commutation relations are

$$\begin{aligned} [l_i^\pm, l_j^\pm] &= -\epsilon_{ijk} l_k^\pm \\ [l_i^\pm, l_j^\mp] &= 0 \end{aligned} \tag{5}$$

so, in the obvious notation

$$\mathfrak{so}(3,1)_{\mathbf{C}} = \mathfrak{so}(3)_{\mathbf{C}} \oplus \mathfrak{so}(3)_{\mathbf{C}} \tag{6}$$

[1] `www.maths.tcd.ie/~houghton/445.html`

[2] There is an i different to our earlier definition, writing $l_i = iJ_i$ gives the previous commutation relation.

[3] Complexifying an algebra is sometimes easier when you think in terms of the abstract algebra rather than a given basis, this is particularly true in the case of $\mathrm{SL}(2,\mathbf{C})$ whose defining representation has basis $\{\sigma_i, i\sigma_i\}$.

and the representations are labeled by two numbers j^+ labeling how the l_i^+ are represented and j^- labeling how the l_i^- are represented.

The fields which are quantized to give quantum particle dynamics in quantum field theory satisfy often Lorentz invariant field equations. The Lorentz action may act on the field as well as on space-time and these (j^+, j^-) label the possible transformation properties of the fields. In other words, the Lorentz transform acts on the fields in one of the matrix representations, the fields are in the carrier space. Massless fermions transform as Weyl spinors $(1/2, 0)$ or conjugate spinors $(0, 1/2)$. Massive fermions mush also transform under a parity transform, this means they are represented by Dirac spinors $(1/2, 0) \oplus (0, 1/2)$. Finally, $(1/2, 1/2)$ is a real representation and this is how a real four-vector transforms.