Rough notes for Maths 445[1]

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1. SO(3)

We will begin by looking at SO(3). There are two obvious ways to define SO(3): it is the group of proper rotation in $\mathbf{R}^3$ and it is the matrix group of special orthogonal 3×3 matrices. An *orthogonal* matrix, A, is one for which

$$AA^T = 1_3 \tag{1}$$

where 1_3 denotes the 3×3 identity matrix $1_3 = \mathrm{diag}(1, 1, 1)$. A *special* matrix is a matrix with unit determinant.

The relationship between proper rotations and special orthogonal matrices can be derived by first typifying rotations as homogeneous[2] length preserving transformations:

$$\begin{array}{rcl} \mathbf{R}^3 & \to & \mathbf{R}^3 \\ \mathbf{x} & \mapsto & \mathbf{x}' = R\mathbf{x} \end{array} \tag{2}$$

such that $|\mathbf{x}| = |\mathbf{x}'|$. Now,

$$|\mathbf{x}'|^2 = \mathbf{x}'^T\mathbf{x}' = \mathbf{x}^T R^T R\mathbf{x} \tag{3}$$

and, since this is supposed to hold for all $\mathbf{x} \in \mathbf{R}^3$,

$$RR^T = 1_3 \tag{4}$$

In other words, R is orthogonal.

Orthogonal matrices have determinant one or minus one:

$$1 = \det RR^T = \det R \det R^T = (\det R)^2 \tag{5}$$

We want to restrict ourselves to proper rotations. Rather than prove that proper rotations correspond to special matrices, we'll just define *proper* rotations as those rotations that correspond to special matrices. What this is doing is excluding space inversion $\sigma = \mathrm{diag}(-1, -1, -1)$ from our definition of rotations. This is sensible for two reasons, firstly, inversion is clearly different to proper rotation in the practical sense that you can rotate modest sized objects, but you can't invert them. It is also useful topologically[3] because the full group of orthogonal matrices O(3) is not path-wise connected, there is no path in the space or orthogonal matrices that connects 1_3 and σ, there can't be because the sign of the determinant would have to change at some point along the path.

[1] `www.maths.tcd.ie/~houghton/445.html`

[2] An inhomogenous transformation is one like $\mathbf{x} \mapsto \mathbf{x} + \mathbf{b}$.

[3] For the moment I use the word topological in the colloquial way: to mean those properties which do not change under continuous deformation.

Before going on we should verify that SO(3) is a group, this means that multiplication must be closed and associative, there must be an identity element and every element of the group must have an inverse. Consider R_1 and R_2 elements of SO(3).

$$R_1R_2(R_1R_2)^T = R_1R_2R_2^TR_1^T = R_1R_1^T = 1_3 \tag{6}$$

and

$$\det R_1R_2 = \det R_1 \det R_2 = 1 \tag{7}$$

so $R_1R_2 \in$SO(3). This means multiplication is closed. It is clearly associative because matrix multiplication is associative. 1_3 is special orthogonal and is an identity element and if $R \in$SO(3) so is R^T and this provides an inverse.

There are two things to note at this point. Firstly, SO(3) is *nonAbelian*, $R_1R_2 \neq R_2R_1$ in general, it is easy to see this by imagining what happen when you rotate an object. The second thing is that SO(3) is not a finite group, it does not have a finite number of elements. However, there is a finite-dimensional space of rotations, this space has interesting properties and we will examine it now.

1.1 Angle and axis parameterization

There are two common parameterizations for SO(3), *the Euler angle parameterization* which we won't look at and the *Angle and axis* parameterization. In the angle action parameterization a rotation is specified by giving the axis of rotation and the angle of rotation about that axis.[4]. An axis is specified by a unit vector $\mathbf{n}$ so we match a rotation to $(\mathbf{n}, \theta)$. There is an immediate complication, the identity transformation corresponds to $(\mathbf{n}, 0)$ for any $\mathbf{n}$. The way around this is to parameterize a rotation by a vector $\theta\mathbf{n}$. In this parameterization no rotation corresponds to the zero vector and, for example, a rotation of $\pi/3$ about the x-axis corresponds to $(\pi/3, 0, 0)$. Explicitly, a rotation of θ about $\mathbf{n} = (n_1, n_2, n_3)$ has matrix form

$$[R_{\mathbf{n}}(\theta)]_{ij} = \epsilon_{ijk}n_k \sin\theta + [\delta_{ij} - n_in_j]\cos\theta + n_in_j. \tag{8}$$

The next question is to decide what the space of rotations is by deciding what the range of θ is. Since a rotation of $\theta > \pi$ is equivalent to a rotation of $\theta - \pi$ in the opposite direction, $0 \leq \theta \leq \pi$. There is a complication though $\pi\mathbf{n}$ and $-\pi\mathbf{n}$ correspond to the same rotation. This means that the space of rotations is a three-ball of radius π with antipodal points on the surface identified. Thus, SO(3) may not be a finite group, but it has a finite-dimensional group space, topologically this space is the three-ball with opposite points on the surface identified. This space is a projective space, the projective three-sphere in fact, and has non-trivial properties.

Consider a loop in the three-ball, if the loop doesn't touch the boundary it can be continuously deformed to a point. If it does touches the boundary at one pair of antipodal point, it can't be deformed to a point. This is illustrated in Fig. 1. A loop that touches the

[4]Rotations are always taken to be counter clockwise

boundary at two pairs of antipodal points can be deformed to a point, this is illustrated in Fig.2. Thus it is in the same class under deformation as the loop that never touches the boundary, but the loop that touches the boundary once is in a separate class.

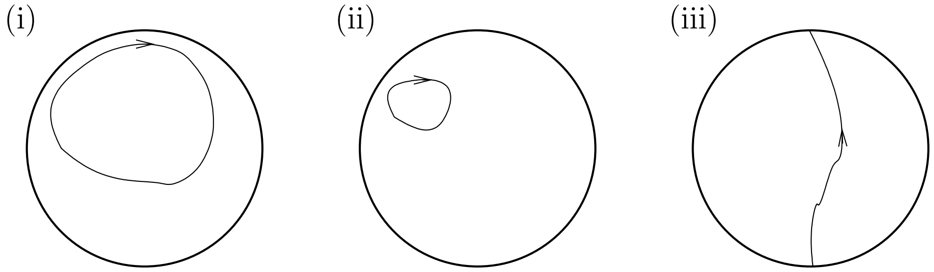

Figure 1: The loops in (i) and (ii) can be deformed into each other and indeed can be deformed continuously to a point. However, the loop in (iii) is of a different type and cannot be deformed continuously to a point.

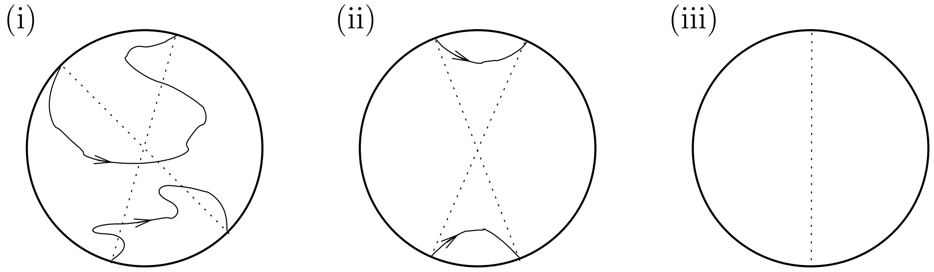

Figure 2: A loop which touches the boundary at two different points can be deformed to a point.

This means there are two types of loops, loops that can be deformed to a point and loops that cannot. Imagine rotating an object through 2π. As you do the rotation you trace out a map in group space. If you rotate the object all the way through 2π you have traced out a loop because rotating something through 2π is the same as not rotating it at all. However, the path cannot be deformed to a point. Thus, by looking at the object you can't tell if it has been rotated or not, but by looking a the path you can. Now rotate it another 2π, now, once a again the object is back where it started, but not the path is also trivial, it touches the boundary of the sphere at two identified points and can be deformed to a point.