

# A geometrical meaning of the Riemann tensor: parallel transport around a closed loop.<sup>1</sup>

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In flat space, if you parallel transport a vector around a closed loop it always comes back to itself. This isn't true in curved space, the purpose of this calculation is to show that the change in a vector after it is transported around a closed loop is related to the Riemann tensor. The calculation is done for a small loop beginning and ending at a point  $p$  and we find that the leading order change to the vector depends on the Riemann curvature at  $p$ . The calculation would be much more difficult for a loop that wasn't small, the change would involve integrating a complicated function around the loop. By taking the loop to be small we can expand all our quantities around their value at  $p$ .

So, consider a point  $p$  with coördinates  $X^a$ . Let

$$x^a(t) = X^a + \epsilon z^t \quad (1)$$

with  $\epsilon \ll 1$  and  $z^a(0) = z^a(T) = 0$  be a small loop beginning and ending at  $p$ . If  $V^a$  is a vector at  $p$  then the parallel transported vector  $V^a(t)$  satisfies the parallel transport equation

$$U^a \nabla_a V^b(t) = 0 \quad (2)$$

with, as usual,

$$U^a = \frac{dx^a}{dt} \quad (3)$$

and  $V^a(0) = V^a$ . From now on we will use quantities without an argument to mean the same quantity at the start of the loop at  $p$ , that is with  $t = 0$ .

Writing the parallel transport equation (2) out more explicitly

$$\frac{dV^a(t)}{dt} + \Gamma_{bc}^a(t) U^b(t) V^c(t) = 0 \quad (4)$$

where these quantities all depend on  $t$  through their spatial dependence. This can be integrated to give

$$V^a(t) - V^a = - \int_0^t dt' \Gamma_{bc}^a(t') U^b(t') V^c(t') \quad (5)$$

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Furthermore, we know

$$U^b(t) = \epsilon \frac{dz^b(t)}{dt} \quad (6)$$

This means that the integrand in (5) is order  $\epsilon$ , and, if we want the leading order expression for  $V^a(t)$  we only need the zeroth order expressions for  $\Gamma_{bc}^a(t')$  and  $V^c(t')$  in the integrand. In other words, by Taylor expansion

$$\begin{aligned} \Gamma_{bc}^a(t') &= \Gamma_{bc}^a + O(\epsilon) \\ V^a(t') &= V^a + O(\epsilon). \end{aligned} \quad (7)$$

Putting this into the equation gives

$$V^a(t) - V^a = -\epsilon \Gamma_{bc}^a V^c \int_0^t dt' \frac{dz^b(t')}{dt'} = -\epsilon \Gamma_{bc}^a V^c z^b(t) + O(\epsilon^2) \quad (8)$$

Hence, using  $z^b(T) = 0$

$$V^a(T) = V^a + O(\epsilon^2) \quad (9)$$

This means that the vector is the same after to parallel transport, to the second order in  $\epsilon$ .

We now calculate the next order in  $\epsilon$ . We can do this because we have now worked out  $V^a(t)$  to first order. Hence, by Taylor expansion

$$\Gamma_{bc}^a(t') = \Gamma_{bc}^a + \epsilon z^d \Gamma_{bc,d}^a + O(\epsilon^2) \quad (10)$$

and, by the above

$$V^a(t') = V^a - \epsilon \Gamma_{bc}^a V^c \int_0^{t'} dt' \frac{dz^b(t')}{dt'} = -\epsilon \Gamma_{bc}^a V^c z^b(t') + O(\epsilon^2) \quad (11)$$

Substituting this into the equation for  $V^a(t)$  (5) we get

$$V^a(t) - V^a = -\epsilon \int_0^t dt' [\Gamma_{bc}^a + \epsilon \Gamma_{bc,d}^a z^d(t')] \frac{dz^b(t')}{dt'} [V^c - \epsilon \Gamma_{ef}^c V^e z^f(t')] + O(\epsilon^3) \quad (12)$$

If we evaluate at  $t = T$  the order  $\epsilon$  part vanishes and we get

$$\begin{aligned} V^a(T) - V^a &= \epsilon^2 \int_0^T dt' \left[ \Gamma_{bc}^a \frac{dz^b(t')}{dt'} \Gamma_{ef}^c V^e z^f(t) - \Gamma_{bc,d}^a z^d(t) \frac{dz^b(t')}{dt'} V^c \right] + O(\epsilon^3) \\ &= \epsilon^2 [\Gamma_{bc}^a \Gamma_{cd}^e - \Gamma_{bc,d}^a] V^c \int_0^T dt' \frac{dz^b(t')}{dt'} z^d(t') + O(\epsilon^3) \end{aligned} \quad (13)$$

where, for convenience, we have renamed some of the summed indices. Next, we use integration by parts and the boundary conditions  $z^a(0) = z^a(T) = 0$  to show

$$\int_0^T dt' \frac{dz^b(t')}{dt'} z^d(t') = - \int_0^T dt' \frac{dz^d(t')}{dt'} z^b(t'). \quad (14)$$

This allows us to antisymmetrize the indices  $b$  and  $d$  in the expression for  $V^a(T) - V^a$  (??):

$$V^a(T) - V^a = \epsilon^2 \frac{1}{2} [\Gamma_{be}^a \Gamma_{cd}^e - \Gamma_{de}^a \Gamma_{cb}^e - \Gamma_{bc,d}^a + \Gamma_{dc,b}^a] V^c \int_0^t dt' \frac{dz^b(t')}{dt'} z^d(t') + O(\epsilon^3) \quad (15)$$

Now, we know

$$R_{abc}{}^d = \Gamma_{ac,b}^d - \Gamma_{bc,a}^d + \Gamma_{ac}^f \Gamma_{bf}^d - \Gamma_{bc}^f \Gamma_{af}^d + O(\epsilon^3) \quad (16)$$

so, with zero torsion,

$$V^a(T) - V^a = \epsilon^2 \frac{1}{2} R_{dcb}{}^a V^c \int_0^t dt' \frac{dz^b(t')}{dt'} z^d(t') \quad (17)$$

Thus, to leading order the change in the vector depends on the Riemann tensor at  $p$  and on an integral factor which does not depend on the derivative of the metric. The integral factor is related to the size of the loop; it is easy to see that it is the area of the loop if the loop is a parallelogram.

Don't worry too much about which of the indices of the Riemann tensor is raised, this can always be moved around using the symmetries of the Riemann tensor:

$$R_{dcb}{}^a = g^{ae} R_{dcbe} = g^{ae} R_{bedc} = g^{ae} R_{ebcd} = R_{bcd}{}^a \quad (18)$$