A geometrical meaning of the Riemann tensor: parallel transport around a closed loop.[1]

6 November 2002

In flat space, if you parallel transport a vector around a closed loop it always comes back to itself. This isn't true in curved space, the purpose of this calculation it to show that the change in a vector after it is transported around a closed loop is related to the Riemann tensor. The calculation is done for a small loop begining and ending at a point p and we find that the leading order change to the vector depends on the Riemann cuvature at p. The calcuation would be much more difficult for a loop that wasn't small, the change would involve integrating a complicated function around the loop. By taking the loop to be small we can expand all our quantities around their value at p.

So, consider a point p with coördinates X^a. Let

$$x^a(t) = X^a + \epsilon z^t \tag{1}$$

with $\epsilon \ll 1$ and $z^a(0) = z^a(T) = 0$ be a small loop begining and ending at p. If V^a is a vector at p then the parallel transported vector $V^a(t)$ satisfies the parallel transport equation

$$U^a \nabla_a V^b(t) = 0 \tag{2}$$

with, as usual,

$$U^a = \frac{dx^a}{dt} \tag{3}$$

and $V^a(0) = V^a$. From now on we will use quantities without an arguement to mean the same quantity at the start of the loop at p, that is with $t = 0$.

Writing the parallel transport equation (??) out more explicitly

$$\frac{dV^a(t)}{dt} + \Gamma^a_{bc}(t)U^b(t)V^c(t) = 0 \tag{4}$$

where these quantities all depend on t through their spatial dependence. This can be integrated to give

$$V^a(t) - V^a = -\int_0^t dt' \Gamma^a bc(t')U^b(t')V^c(t') \tag{5}$$

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Furthermore, we know

$$U^b(t) = \epsilon \frac{dz^b(t)}{dt} \tag{6}$$

This means that the integrand in (**??**) is order ϵ, and, if we want the leading order expression for $V^a(t)$ we only need the zeroth order expressions for $\Gamma^a_{bc}(t')$ and $V^c(t')$ in the integrand. In other words, by Taylor expansion

$$\begin{aligned} \Gamma^a_{bc}(t') &= \Gamma^a_{bc} + O(\epsilon) \\ V^a(t') &= V^a + O(\epsilon). \end{aligned} \tag{7}$$

Putting this into the equation gives

$$V^a(t) - V^a = -\epsilon\Gamma^a bc V^c \int_0^t dt' \frac{dz^b(t')}{dt'} = -\epsilon\Gamma^a bc V^c z^b(t) + O(\epsilon^2) \tag{8}$$

Hence, using $z^b(T) = 0$

$$V^a(T) = V^a + O(\epsilon^2) \tag{9}$$

This means that the vector is the same after to parallel transport, to the second order in ϵ.

We now calculate the next order in ϵ. We can do this because we have now worked out $V^a(t)$ to first order. Hence, by Taylor expansion

$$\Gamma^a_{bc}(t') = \Gamma^a_{bc} + \epsilon z^d \Gamma^a_{bc,d} + O(\epsilon^2) \tag{10}$$

and, by the above

$$V^a(t') = V^a - \epsilon\Gamma^a bc V^c \int_0^t dt' \frac{dz^b(t')}{dt'} = -\epsilon\Gamma^a bc V^c z^b(t') + O(\epsilon^2) \tag{11}$$

Substituting this into the equation for $V^a(t)$ (**??**) we get

$$V^a(t) - V^a = -\epsilon \int_0^t dt' [\Gamma^a_{bc} + \epsilon\Gamma^a_{bc,d} z^d(t)] \frac{dz^b(t')}{dt'} [V^c - \epsilon\Gamma^c_{ef} V^e z^f(t)] + O(\epsilon^3) \tag{12}$$

If we evaluate at $t = T$ the order ϵ part vanishes and we get

$$\begin{aligned} V^a(T) - V^a &= \epsilon^2 \int_0^t dt' \left[\Gamma^a_{bc} \frac{dz^b(t')}{dt'} \Gamma^c_{ef} V^e z^f(t) - \Gamma^a_{bc,d} z^d(t) \frac{dz^b(t')}{dt'} V^c \right] + O(\epsilon^3) \\ &= \epsilon^2 \left[\Gamma^a_{be}\Gamma^e_{cd} - \Gamma^a_{bc,d} \right] V^c \int_0^t dt' \frac{dz^b(t')}{dt'} z^d(t') + O(\epsilon^3) \end{aligned} \tag{13}$$

where, for convenience, we have renamed some of the summed indices. Next, we use integration by parts and the boundary conditions $z^a(0) = z^a(T) = 0$ to show

$$\int_0^t dt' \frac{dz^b(t')}{dt'} z^d(t') = -\int_0^t dt' \frac{dz^d(t')}{dt'} z^b(t'). \tag{14}$$

This allows us to antisymmetrize the indices b and d in the expression for $V^a(T) - V^a$ (**??**):

$$V^a(T) - V^a = \epsilon^2 \frac{1}{2} \left[\Gamma^a_{be}\Gamma^e_{cd} - \Gamma^a_{de}\Gamma^e_{cb} - \Gamma^a_{bc,d} + \Gamma^a_{dc,b} \right] V^c \int_0^t dt' \frac{dz^b(t')}{dt'} z^d(t') + O(\epsilon^3) \quad (15)$$

Now, we know

$$R_{abc}{}^d = \Gamma^d_{ac,b} - \Gamma^d_{bc,a} + \Gamma^f_{ac}\Gamma^d_{bf} - \Gamma^f_{bc}\Gamma^d_{af} + O(\epsilon^3) \quad (16)$$

so, with zero torsion,

$$V^a(T) - V^a = \epsilon^2 \frac{1}{2} R_{dcb}{}^a V^c \int_0^t dt' \frac{dz^b(t')}{dt'} z^d(t') \quad (17)$$

Thus, to leading order the change in the vector depends on the Riemann tensor at p and on an integral factor which does not depend on the derivative of the metric. The integral factor is related to the size of the loop; it is easy to see that it is the area of the loop if the loop is a parallelogram.

Don't worry too much about which of the indices of the Riemann tensor is raised, this can always be moved around using the symmetries of the Riemann tensor:

$$R_{dcb}{}^a = g^{ae} R_{dcbe} = g^{ae} R_{bedc} = g^{ae} R_{ebcd} = R^a{}_{bcd} \quad (18)$$