The Bianchi identites.[1]

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The Riemann tensor obeys a Bianchi identity:

$$\nabla_{[e|}R_{ab|cd]} = 0 \tag{1}$$

Since $R_{abcd} = -R_{abdc}$ this expands out to

$$\nabla_e R_{abcd} + \nabla_c R_{abde} + \nabla_d R_{abec} = 0 \tag{2}$$

This is easily proved in normal coördinate where, at a given point, the connection coefficients can be set to zero: in normal coördinates

$$\nabla_e R_{abcd} + \nabla_c R_{abde} + \nabla_d R_{abec} = \partial_e R_{abcd} + \partial_c R_{abde} + \partial_d R_{abec} \tag{3}$$

and then use the normal coördinate formula

$$R_{abcd} = \frac{1}{2}\left(-g_{ac,bd} + g_{ad,bc} - g_{bc,ad} - g_{bd,ac}\right) \tag{4}$$

We can derive further identities, called contracted Bianchi identities: multiplying by g^{ac} gives

$$\nabla_e g^{ac} R_{abcd} + g^{ac}\nabla_c R_{abde} + \nabla_d g^{ac} R_{abec} = 0 \tag{5}$$

Now

$$g^{ac} R_{abcd} = R_{bd} \tag{6}$$

the Ricci tensor and

$$g^{ac} R_{abec} = -g^{ac} R_{abce} = R_{be} \tag{7}$$

so

$$\nabla_e R_{bd} + \nabla^a R_{abde} - \nabla_d R_{be} = 0 \tag{8}$$

Multiplying again by g^{be} we have

$$g^{be}\nabla_e R_{bd} + \nabla^a g^{be} R_{abde} - \nabla_d g^{be} R_{be} = 0 \tag{9}$$

or

$$2\nabla^b R_{bd} - \nabla_d R = 0 \tag{10}$$

This is sometimes written as

$$\nabla^b G_{bd} = 0 \tag{11}$$

where

$$G_{bd} = R_{bd} - \frac{1}{2} R g_{bd} \tag{12}$$

is the Einstein tensor.

[1] Conor Houghton, houghton@maths.tcd.ie please send me any corrections.