Independent components of the Riemann tensor[1]

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So, how many independent components has the Riemann tensor in d-dimensional spacetime? The Riemann tensor is a $(0,4)$ tensor with three symmetries

$$\begin{aligned} R_{abcd} &= -R_{bacd} \\ R_{abcd} &= R_{cdab} \\ R_{abcd} &= -R_{abdc} \end{aligned} \tag{1}$$

and satisfying the cyclic identity

$$R_{abcd} + R_{acdb} + R_{adbc} = 0 \tag{2}$$

Let us begin by ignoring the cyclic identity and counting the independent components left after taking the symmetries in Eq. 1 into account. First concentrate on the ab indices. Since these are skew-symmetric they label $n = d(d-1)/2$ independant components. For convenience let us label these components with $I = 1, 2, \dots, n$. Next, the cd indices have the same symmetry and label the same number of components. Let us label these componets with $J = 1, 2, \dots, n$. Finally, R_{IJ} is symmetric in its indices and therefore has $n(n+1)/2$ independant components with

$$\frac{1}{2}n(n+1) = \frac{1}{4}d(d-1)\left[\frac{1}{2}d(d-1)+1\right]. \tag{3}$$

Now, we take the cyclic identity into account. The difficulty here is deciding how many independant cyclic identities there are. The easiest way to do this is to see that the cyclic identity is equivalent to

$$R_{[abcd]} = 0. \tag{4}$$

To see this, note that, under the symmetries in Eq. 1

$$8R_{abcd} = R_{abcd} - R_{abdc} + R_{badc} - R_{bacd} + R_{cdab} - R_{dcab} + R_{dcba} - R_{cdba}. \tag{5}$$

There are similar expression for $8R_{acdb}$ and $8R_{adbc}$. If each term in the cyclic identity is expanded out in this way the resulting 24 term identity is Eq. 4. Now, there is an identity of this form for any choice of four distinct unordered indices, hence, the number of independant constraints coming from the cyclic identity is d choose four. This means that the total number of independant components of the Riemann tensor $N(d)$ is given by

$$N(d) = \frac{1}{4}d(d-1)\left[\frac{1}{2}d(d-1)+1\right] - \binom{d}{4} \tag{6}$$

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Algebraic manipulation show that

$$N(d) = \frac{1}{12}d^2(d^2-1) \tag{7}$$

One reason this is interesting is that it shows the Riemann tensor can be expressed in terms of the Ricci scalar in two dimensions and in terms of the Ricci tensor in three dimensions. To see this, note that $N(2) = 1$ and $N(3) = 6$ and remember that the Ricci tensor is a symmetric $(0,2)$ tensor and so has six independant components in three dimensions. However, in four dimensions $N(4) = 20$ whereas the Ricci tensor has only ten independant components. In fact, there is a tensor, called the Weyl tensor W_{abcd}, which is defined in terms of Riemann tensor, has the same symmetries as the Riemann tensor, but has the additional property that it is trace free:

$$g^{ab}W_{bcde} = 0 \tag{8}$$

In four dimensions the Weyl tensor has ten independant components and can be thought of as containing the information in the Riemann tensor that is not in the Ricci tensor or Ricci scalar. It is discuss at (d'I 87).

In two and three dimensions it is easy to write down formulae explicitly for the Riemann tensor. This is because the Riemann tensor must reduce to Ricci tensor and then the Ricci scalar under contraction of its indices with the metric tensor. In two dimensions, this means that the Riemann tensor is a tensor depending only on the Ricci scalar and the metric, reducing to the Ricci scalar under contraction. The only expression with the correct symmetries is

$$R_{abcd} = \alpha R(g_{ac}g_{bd} - g_{ad}g_{cd}) \tag{9}$$

and contracting on both sides with $g^{ac}g^{bd}$ shows that $\alpha = 1/2$. In three dimensions there are two combinations with the correct symmetry, so

$$R_{abcd} = \alpha(R_{[a|c}g_{b|d]} + R_{[b|d}g_{a|c]}) + \beta R g_{[a|c}g_{b|d]} \tag{10}$$

where α and β are constants to be determined by requiring that $R_{bd} = R^a{}_{bad}$. Expanding out the skew-symmetrizations we get

$$R_{abcd} = \frac{\alpha}{2}(R_{ac}g_{bd} - R_{dc}g_{ba} + R_{bd}g_{ac} - R_{cd}g_{ab}) + \frac{\beta}{2}R(g_{ac}g_{bd} - g_{dc}g_{ba}) \tag{11}$$

and so

$$\begin{aligned} R_{bd} = g^{ac}R_{abcd} &= \frac{\alpha}{2}(Rg_{bd} - R_{bd} + dR_{bd} - R_{bd}) + \beta R g_{bd}(d-1) \\ &= \left[-\frac{\alpha}{2} + (d-1)\beta\right] R g_{bd} + \frac{(d-2)\alpha}{2}R_{bd} \end{aligned} \tag{12}$$

and hence

$$\begin{aligned}
\alpha &= \frac{2}{d-2} \\
\beta &= \frac{1}{(d-2)(d-1)}
\end{aligned} \tag{13}$$