The explicit formula for the Riemann tensor[1]

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This calculation isn't too bad when you actually sit down and do it. First, by definition

$$\nabla_a(\nabla_b U_c) - \nabla_b(\nabla_a U_c) = R_{abc}{}^d U_d. \tag{1}$$

Now $\nabla_b U_c$ is a $(0,2)$ tensor, so

$$\nabla_a(\nabla_b U_c) = (\nabla_b U_c)_{,a} - \Gamma^d_{ab}\nabla_d U_c - \Gamma^d_{ac}\nabla_b U_d \tag{2}$$

Next, we know

$$\nabla_b U_c = U_{c,b} - \Gamma^d_{bc} U_d \tag{3}$$

so

$$\nabla_a(\nabla_b U_c) = U_{c,a,b} - \Gamma^d_{bc,a}U_d - \Gamma^d_{bc}U_{d,a} - \Gamma^d_{ab}U_{c,d} + \Gamma^d_{ab}\Gamma^e_{dc}U_e - \Gamma^d_{ac}U_{d,b} + \Gamma^d_{ac}\Gamma^e_{bd}U_e \tag{4}$$

and moving terms around this is

$$\nabla_a(\nabla_b U_c) = U_{c,a,b} - \Gamma^d_{ab}U_{c,d} + \Gamma^d_{ab}\Gamma^e_{dc}U_e - \Gamma^d_{bc}U_{d,a} - \Gamma^d_{ac}U_{d,b} - \Gamma^d_{bc,a}U_d + \Gamma^e_{ac}\Gamma^d_{be}U_d \tag{5}$$

We can see that the first three terms are symmetric in a and b on there own and the fifth and fourth term are symmetric in a and b together. Hence

$$\nabla_a(\nabla_b U_c) - \nabla_b(\nabla_a U_c) = -\Gamma^d_{bc,a}U_d + \Gamma^d_{ac,b}U_d + \Gamma^e_{ac}\Gamma^d_{be}U_d - \Gamma^e_{bc}\Gamma^d_{ae}U_d. \tag{6}$$

Hence, by arbitrarity of U_d we get

$$R_{abc}{}^d = \Gamma^d_{ac,b} - \Gamma^d_{bc,a} + \Gamma^e_{ac}\Gamma^d_{be} - \Gamma^e_{bc}\Gamma^d_{ae} \tag{7}$$

as required.

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