The Sawtooth function

9 November 2003

This used to be part of a problem sheet, this is why it is laid out in problem sheet format.

1. Sketch the two sawtooth and periodic sawtooth functions described in the next question.

Solution: The sawtooth

and the periodic sawtooth

2. Find the Laplace transform of the saw tooth function

\[ f(t) = \begin{cases} 
  t & 0 \leq t < 1 \\
  0 & t \geq 1 
\end{cases} \]  

(1)

Next, find the Laplace transform of the periodic saw tooth function with period one given by

\[ f(t) = t \quad 0 \leq t < 1 \]

\[ f(t+1) = f(t) \]  

(2)

Solution: From the definition of the Laplace transform and integrating by parts

\[ \mathcal{L}(f) = \int_0^\infty f(t)e^{-st} \, dt = \int_0^1 te^{-st} \, dt \]

\[ = \left[ -\frac{t}{s}e^{-st} \right]_t=0^1 + \int_0^1 \frac{1}{s} e^{-st} \, dt \]

\[ = \frac{1}{s} e^{-s} + 0 + \left[ \frac{1}{s^2} e^{-st} \right]_t=0^1 \]

\[ = \frac{1}{s^2} - \frac{1}{s} e^{-s} - \frac{1}{s^2} e^{-s} \]

We know the Laplace transform of the periodic function is

\[ \mathcal{L}(f) = \frac{1}{1 - e^{-s}} \int_0^1 f(t)e^{-st} \, dt \]  

(3)

and this integral is identical to the one we just did in the previous question. So the answer for the periodic saw tooth is

\[ \frac{1}{1 - e^{-s}} \left( \frac{1}{s^2} - \frac{1}{s} e^{-s} - \frac{1}{s^2} e^{-s} \right) \]  

(4)