The Sawtooth function

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This used to be part of a problem sheet, this is why it is laid out in problem sheet format.

1. Sketch the two sawtooth and periodic sawtooth functions described in the next question.
   
   Solution: The sawtooth

   ![Graph of the sawtooth function](image1)

   and the periodic sawtooth

   ![Graph of the periodic sawtooth function](image2)

2. Find the Laplace transform of the saw tooth function

   \[ f(t) = \begin{cases} t & 0 \leq t < 1 \\ 0 & t \geq 1 \end{cases} \]  

   (1)

   Next, find the Laplace transform of the periodic saw tooth function with period one given by

   \[ f(t) = t \quad 0 \leq t < 1 \]

   \[ f(t + 1) = f(t) \]  

   (2)
Solution: From the definition of the Laplace transform and integrating by parts

\[
\mathcal{L}(f) = \int_0^\infty f(t)e^{-st} \, dt = \int_0^1 te^{-st} \, dt
\]

\[
= \left[ -\frac{t}{s}e^{-st} \right]_0^1 + \int_0^1 \frac{1}{s}e^{-st} \, dt
\]

\[
= -\frac{1}{s}e^{-s} + 0 + \left[ \frac{1}{-s^2}e^{-st} \right]_0^1
\]

\[
= \frac{1}{s^2} - \frac{1}{s}e^{-s} - \frac{1}{s^2}e^{-s}
\]

We know the Laplace transform of the periodic function is

\[
\mathcal{L}(f) = \frac{1}{1 - e^{-s}} \int_0^1 f(t)e^{-s} \, dt
\] (3)

and this integral is identical to the one we just did in the previous question. So the answer for the periodic saw tooth is

\[
\frac{1}{1 - e^{-s}} \left( \frac{1}{s^2} - \frac{1}{s}e^{-s} - \frac{1}{s^2}e^{-s} \right)
\] (4)