Brief note on eigenvectors and eigenvalues[1]

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A matrix A has eigenvalue λ and eigenvector $\mathbf{x}$ if[2]

$$A\mathbf{x} = \lambda\mathbf{x} \tag{1}$$

For this to make sense the length of $\mathbf{x}$ must not be zero. For example one of the the eigenvectors of

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \tag{2}$$

is

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{3}$$

with corresponding eigenvalue 3. In fact, the matrix has two eigenvalues, the other one is -1 and the corresponding eigenvector is

$$\mathbf{x}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{4}$$

The general thing is that a $n \times n$ matrix has n-eigenvectors.

One thing to notice is that an eigenvector can have any length. This is easy to see, say $\mathbf{x}$ is an eigenvector of A corresponding to λ, then, consider $\mu\mathbf{x}$, the vector you get by multiplying $\mathbf{x}$ by some nonzero number μ. Well

$$A(\mu\mathbf{x}) = \mu A\mathbf{x} = \mu\lambda\mathbf{x} = \lambda(\mu\mathbf{x}) \tag{5}$$

so $\mu\mathbf{x}$ is also an eigenvector corresponding to the same eigenvalue. Taking the example above you could consider

$$2\mathbf{x}_1 = 2\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \tag{6}$$

By multiplying by the matrix you can see that this is also an eigenvector with eigenvalue 3.

The way to calculate the eigenvalues of a matrix is to write down the characteristic equation. This works like this: if $\mathbf{x}$ is an eigenvector with eigenvalue λ then, as before,

$$A\mathbf{x} = \lambda\mathbf{x} \tag{7}$$

[1]Conor Houghton, `houghton@maths.tcd.ie` please send me any corrections. These notes are intended to be helpful, feel free to ignore them.

[2]It is very common in print to write vectors in bold: in a book the vector is written as $\mathbf{x}$ but because it is hard to do bold in handwriting it is normal to handwrite $\underline{x}$. Most people don't use bold for matrices, Kreyszig does. It isn't so important, but it is often confusing if you don't use bold or underlining for vectors.

or, put another way

$$(A - \lambda\mathbf{1})\mathbf{x} = 0 \tag{8}$$

where $\mathbf{1}$ is the identity matrix, it is the matrix with ones on the diagonal and zeros everywhere else. Now, if, say you have a matrix, B, with $\det B \neq 0$ then there is another matrix B^{-1} so that $BB^{-1} = \mathbf{1}$. In the case of 2×2 matrices you have a formula for the inverse, if

$$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{9}$$

then the inverse is

$$B^{-1} = \frac{1}{\det B}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \tag{10}$$

There are similar, but more complicated, formula for larger matrices. Now, if you have matrix equation

$$B\mathbf{x} = 0 \tag{11}$$

you can multiply across by B^{-1} to get

$$B^{-1}B\mathbf{x} = \mathbf{1}\mathbf{x} = \mathbf{x} = 0 \tag{12}$$

In other words, the matrix equation $B\mathbf{x} = 0$ implies $\mathbf{x} = 0$ if B has nonzero determinant. Now, we want the eigenvector to be a nonzero vector so the matrix $A - \lambda\mathbf{1}$ multiplying $\mathbf{x}$ in the eigenvector equation

$$(A - \lambda\mathbf{1})\mathbf{x} = 0 \tag{13}$$

must have a zero determinant. It must happen that

$$\det(A - \lambda\mathbf{1}) = 0. \tag{14}$$

This is the **characteristic equation**,[3] it is the equation for the eigenvalues. If A is 2×2 it is a quadratic equation, for a $n \times n$ matrix it is a degree n polynomial.[4] In this way, a $n\times n$ matrix will generally have n eigenvalues but, as we have seen, sometimes it has fewer because the characteristic equation has repeated roots.

As an example consider

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \tag{15}$$

as above. The characteristic equation is

$$\det(A - \lambda\mathbf{1}) = \det\begin{pmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix} = 0. \tag{16}$$

[3]Sometime the notation $\chi(A)$, or even, $\chi(A, \lambda)$, is used for $\det(A - \lambda\mathbf{1})$. χ is the greek letter *chi* so it is often used to denote things beginning with *ch* where the *ch* is pronounces with a *k* sound.

[4]A degree n polynomial is one where the highest power is n.

Or, using the other notation for determinant, $\det A = |A|$ the characteristic equation is

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = (1-\lambda)(1-\lambda) - 4 = 0. \tag{17}$$

We can factorize the quadratic

$$(1-\lambda)(1-\lambda) - 4 = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1) = 0 \tag{18}$$

and so $\lambda = 3$ or $\lambda = -1$ are the eigenvalues.

Find the eigenvectors is easy once you know the eigenvalues. All you do is write down the equation and solve it. Take the example above, let the $\lambda = 3$ eigenvector be

$$\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix} \tag{19}$$

where a and b are unknown, then substitute in to get

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 3 \begin{pmatrix} a \\ b \end{pmatrix} \tag{20}$$

and so multiplying it out gives

$$\begin{pmatrix} a + 2b \\ 2a + b \end{pmatrix} = \begin{pmatrix} 3a \\ 3b \end{pmatrix}. \tag{21}$$

and so

$$\begin{aligned} a + 2b &= 3a \\ 2a + b &= 3b \end{aligned} \tag{22}$$

This looks like two equations, but it isn't. The two equations are both the same and give

$$a = b \tag{23}$$

This is the way it has to be because the matrix $A - \lambda\mathbf{1}_2$ has zero determinant.[5] Now the equation tells you what a is in terms of b but doesn't tell you how long $\mathbf{x}$ is. This is what we expect, since an eigenvector can be any nonzero length. Here

$$\mathbf{x} = 0. \tag{24}$$

[5] If a matrix has zero determinant it means that its two rows are linearly related, in other words, proportional to each other. If $B\mathbf{x} = 0$ this means that $\mathbf{x}$ is perpendicular to the rows of B, so if the rows are all independent $\mathbf{x}$ must be zero. If the determinant of B is zero then the rows are not independent and so, in the 2×2 case, the vector $\mathbf{x}$ must just be perpendicular to the rows. This specifies a direction but not a length.

Don't be worried if the equation just says one of the components is zero. An example of this is the $\lambda = 2$ eigenvector of

$$\begin{pmatrix} 2 & -1 \\ 0 & -4 \end{pmatrix} \tag{25}$$

where the equation is

$$\begin{pmatrix} 2 & -1 \\ 0 & -4 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 2 \begin{pmatrix} a \\ b \end{pmatrix} \tag{26}$$

and so

$$2a - b = 2a \tag{27}$$

or $b = 0$. In other words you know that $b = 0$ but a can be anything, except zero. It cannot be zero because an eigenvector has nonzero length. $a = 1$ would be a good choice.

Summary

So, in brief summary, to work out the eigenvalues and eigenvectors of a matrix A, you start by writing out the characteristic equation:

$$\det(A - \lambda\mathbf{1}) = 0 \tag{28}$$

To do this you need to recall the formula for the determinant of a matrix. For a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{29}$$

then $\det A = ad - bc$. Next you solve the characteristic equation to find the possible values of λ and finally you work out the eigenvectors by substituting back into the eigenvector equation

$$A\mathbf{x} = \lambda\mathbf{x} \tag{30}$$

so, if A is 2×2, write

$$\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix} \tag{31}$$

and multipling out Eq. 30 this will give an equation for a in term of b.

Reference

A book which describes eigenvectors and eigenvalues in a way which is pertinent to this course is *Linear algebra and its applications* by Gilbert Strang.