Notes on the Z-transform, part 4

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1 Solving difference equations

At the end of note 3 we saw how to solve a difference equation using Z-transforms. Here is a similar example, solve

\[ x_{k+2} - 6x_{k+1} - 55x_k = 0 \]  

(1)

with \( x_1 = 1 \) and \( x_0 = 0 \). We begin by taking the Z-transform of both sides, remember, if we write \( Z[x_k] = X(z) \) then

\[
Z[(x_{k+1})] = zX - 2x_0 \\
Z[(x_{k+2})] = z^2X - z^2x_0 - zx_1
\]

(2)

so, in this case we get

\[ z^2X - z - 6zX - 55X = 0 \]  

(3)

or

\[ X = \frac{z}{z^2 - 6z - 55} \]  

(4)

As before, we do a partial fraction expansion, but, first we move the \( z \) over to the right hand side,

\[
\frac{1}{z}X = \frac{1}{z^2 - 6z - 55} = \frac{1}{(z - 11)(z + 5)} = \frac{A}{z - 11} + \frac{B}{z + 5}
\]

(5)

giving

\[ 1 = A(z + 5) + B(z - 11) \]  

(6)

Choose \( z = 11 \) to learn \( A = 1/16 \) and \( z = -5 \) to learn \( B = -16 \). Therefore,

\[
\frac{1}{z}X = \frac{1}{16(z - 11)} - \frac{1}{16(z + 5)}
\]

(7)

or

\[ X = -\frac{1}{16(z - 11)} + \frac{1}{16(z + 5)} \]  

(8)

Both terms of the form \( z/(z - r) \) and so

\[ x_k = -\frac{1}{16}(11)^k + \frac{1}{16}(-5)^k \]  

(9)

The basic process is simple, you take the Z-transform of both sides, you solve for \( X \), you use partial fractions to put it into a convenient form and then work out \( x_k \). In the rest of this note we will look at a variety of examples which exhibit the various difficulties that might be encountered doing this.

1.1 Not zero on the right hand side

Consider the difference equation

\[ x_{k+2} - 6x_{k+1} - 55x_k = (-3)^k \]  

(10)

with \( x_1 = 0 \) and \( x_0 = 0 \). This is different than the previous example in that the right hand side is not zero, it is \( 3^k \). This doesn’t make such a difference, take the Z-transform of both sides, and noting the trivial initial data \( (x_0 = 0 \text{ and } x_1 = 0) \),

\[ z^2X - 6zX - 55X = Z[(3^k)] = \frac{z}{z + 3} \]  

(11)

thus,

\[ \frac{1}{z}X = \frac{1}{(z - 11)(z + 5)(z + 3)} \]  

(12)

Now, use partial fractions

\[
\frac{1}{(z - 11)(z + 5)(z + 3)} = \frac{A}{z - 11} + \frac{B}{z + 5} + \frac{C}{z + 3}
\]

(13)

or

\[ 1 = A(z + 5)(z + 3) + B(z - 11)(z + 3) + C(z - 11)(z + 5) \]  

(14)

Now, the usual, \( z = 11 \) gives \( A = 1/224 \), \( z = -5 \) gives \( B = 1/32 \) and \( z = -3 \) gives \( C = -1/28 \). Not forgetting the minus in equation (12) this gives

\[ X = \frac{1}{224} \frac{z}{z - 11} - \frac{1}{32} \frac{z}{z + 5} + \frac{1}{28} \frac{z}{z + 3} \]  

(15)

and so

\[ x_k = -\frac{1}{224}(11)^k - \frac{1}{32}(-5)^k + \frac{1}{28}(-3)^k \]  

(16)

Remember that the sequence on the right hand side might be a sequence of ones, \( 1 = 1^k \), so, consider

\[ x_{k+2} - 6x_{k+1} - 55x_k = 1 \]  

(17)

with \( x_1 = 0 \) and \( x_0 = 0 \). Take the Z-transform of both sides

\[ X - 6zX - 55X = Z[(1^k)] = \frac{z}{z - 1} \]  

(18)

and so

\[ \frac{1}{z}X = \frac{1}{(z - 11)(z + 5)(z - 1)} \]  

(19)

Without going through the calculation, the partial fraction expansion is

\[
\frac{1}{(z - 11)(z + 5)(z - 1)} = \frac{1}{160} \frac{1}{z - 11} + \frac{1}{96} \frac{1}{z + 5} + \frac{1}{60} \frac{1}{z - 1}
\]

(20)

and so,

\[ x_k = \frac{1}{160}(11)^k + \frac{1}{96}(-5)^k + \frac{1}{60} \]  

(21)
1.2 Repeated root

As happens with Laplace transforms, there can be a repeated root. Consider

\[ x_{k+2} - 6x_{k+1} - 55x_k = 11^k \]  

(22)

with \( x_1 = 0 \) and \( x_0 = 0 \). Take the \( z \)-transform of both sides

\[ X - 6zX - 55X = Z[(11^k)] = \frac{z}{z-11} \]  

(23)

and so

\[ \frac{1}{z}X = \frac{1}{(z-11)(z+5)} \]  

(24)

Now, remember that for repeated root the partial fraction expansion has a term with the root and a term with its square:

\[ \frac{1}{(z-11)^2(z+5)} = \frac{A}{z-11} + \frac{B}{z-11} + \frac{C}{z+5} \]  

(25)

Thus,

\[ 1 = A(z-11)(z-5) + B(z+5) + C(z-11)^2 \]  

(26)

Now, \( z = 11 \) gives \( B = 1/16 \) and \( z = -5 \) gives \( C = 1/256 \). The problem is that no choice of \( z \) gives \( A \) on its own, instead we chose any value that hasn’t been used before, \( z = 0 \) for example,

\[ 1 = -55A + 5B + 121C \]  

(27)

and now, we substitute for the known values of \( B \) and \( C \),

\[ 1 = -55A + \frac{5}{16} + \frac{121}{256} = -55A + \frac{201}{256} \]  

(28)

Hence

\[ -55A = 1 - \frac{201}{256} = \frac{55}{266} \]  

(29)

so \( A = -1/256 \). This means that

\[ X = -\frac{1}{256} \frac{z}{z-11} + \frac{1}{16} \frac{z}{(z-11)^2} + \frac{1}{256} \frac{z}{z+5} \]  

(30)

The only problem now is that the \( z/(z-11)^2 \) term might look unfamiliar, but recall

\[ Z[(kr^{k-1})] = \frac{z}{(z-r)^2} \]  

(31)

has this form. We get

\[ x_k = -\frac{1}{256} 11^k + \frac{k}{16} 11^{k-1} + \frac{1}{256} (-5)^k \]  

(32)

1.3 Less convenient initial data

So far the values of \( x_0 \) and \( x_1 \) have been chosen to keep things as simple as possible. More general values of \( x_0 \) and \( x_1 \) might be less convenient, but there is no big change in the method. Consider

\[ x_{k+2} - 6x_{k+1} - 55x_k = 0 \]  

(33)

with \( x_1 = 2 \) and \( x_0 = 6 \). Taking the \( Z \)-transform of both sides gives

\[ zX^2 - 2zX_0 - 2zX + 6zX + 36z - 55X = 0 \]  

(34)

and substituting for the initial data

\[ zX^2 - 2z^2 - 2z - 6zX + 36z - 55X = 0 \]  

(35)

Moving everything around, this gives,

\[ X = \frac{6z^2 - 34z}{z^2 - 6z - 55} \]  

(36)

We still want a \( z \) on top when we are finished, so move one over:

\[ \frac{1}{z}X = \frac{6z - 34}{z^2 - 6z - 55} \]  

(37)

and now, remember that the partial fraction expansion works fine provided what on top is a polynomial of degree less than than the polynomial on the bottom, so we have

\[ \frac{6z - 34}{(z-11)(z+5)} = \frac{A}{z-11} + \frac{B}{z+5} \]  

(38)

or

\[ 6z - 34 = A(z+5) + B(z-11) \]  

(39)

Choose \( z = 11 \) to get

\[ 66 - 34 = 16A \]  

(40)

or \( A = 2 \). \( z = -5 \) gives

\[ -30 - 34 = -16B \]  

(41)

or \( B = 4 \). Thus,

\[ X = \frac{2}{z-11} + \frac{4}{z+5} \]  

(42)

and

\[ x_k = 2(11)^k + 4(-5)^k \]  

(43)
1.4 Examples involving the delay theorem

Consider

\[ x_{k+2} - 6x_{k+1} - 55x_k = \delta_k \]  

(44)

with \( x_1 = 0 \) and \( x_0 = 0 \). \( \delta_k \) is the unit pulse

\[ (\delta_k) = (1, 0, 0, 0, \ldots) \]  

(45)

and \( \mathcal{Z}[\delta_k] = 1 \). Hence

\[ z^2X - 6zX - 55X = \mathcal{Z}[\delta_k] = 1 \]  

(46)

so

\[ X = \frac{1}{(z-11)(z+5)} \]  

(47)

Using the partial fraction expansion, this gives

\[ X = \frac{1}{16z-11} - \frac{1}{16z+5} \]  

(48)

The problem now is that there are no \( z \)'s on top. However, if we rewrite it as

\[ X = \frac{1}{z} \left[ \frac{1}{16z-11} - \frac{1}{16z+5} \right] \]  

(49)

Now, the part inside the square brackets has the form we are familiar with, we can see

\[ \mathcal{Z} \left[ \left( \frac{1}{16} \right)^k - \left( \frac{-5}{16} \right)^k \right] = \frac{1}{16z-11} - \frac{1}{16z+5} \]  

(50)

and we also know from the delay theorem that the effect of multiplying by \( 1/z^k \) is to delay the sequence by \( k \) steps. Hence, the sequence here is delayed by one step and

\[ x_k = \begin{cases} 0 & \text{if } z = 0 \\ \frac{1}{16}11^{k-1} - \frac{1}{16}(-5)^{k-1} & \text{if } z \geq 1 \\ \end{cases} \]  

(51)

Of course, the sequence on the right might be more complicated, consider

\[ x_{k+2} - 6x_{k+1} - 55x_k = y_k \]  

(52)

with \( x_1 = 0 \) and \( x_0 = 0 \) where

\[ (y_k) = (0, 2, 0, 0, 0, \ldots) \]  

(53)

We have to calculate \( \mathcal{Z}[(y_k)] \) before we can make any progress. However, it is easy to see that \( (y_k) \) is the first delay of twice the unit pulse \( (y_k) = 2(\delta_{k-1}) \) so

\[ \mathcal{Z}[(y_k)] = \frac{2}{z} \]  

(54)

Thus the Z-transform of the difference equation gives

\[ z^2X - 6zX - 55X = \frac{2}{z} \]  

(55)

so,

\[ X = \frac{2}{z(z-11)(z+5)} \]  

(56)

There are two ways to go on from here, the first is to use the previous partial fractions expansion

\[ X = \frac{2}{z} \left[ \frac{1}{16z-11} - \frac{1}{16z+5} \right] \]  

(57)

\[ = \frac{2}{z} \left[ \frac{1}{16z-11} - \frac{1}{16z+5} \right] \]  

(58)

so now we are dealing with a two step delay and, keeping the extra factor of two in mind

\[ x_k = \begin{cases} 0 & \text{if } z \leq 1 \\ \frac{1}{16}11^{k-2} - \frac{1}{16}(-5)^{k-2} & \text{if } z \geq 2 \\ \end{cases} \]  

(59)

The other way to make progress is to work out the partial fraction expansion

\[ \frac{1}{z(z-11)(z+5)} = \frac{1}{16z-11} - \frac{1}{16z+5} + \frac{1}{80z+5} \]  

(60)

This means that

\[ X = \frac{2}{z} \left[ \frac{1}{16z-11} - \frac{1}{16z+5} \right] \]  

(61)

and using the delay theorem

\[ x_k = \begin{cases} 0 & \text{if } z = 0 \\ -\frac{2}{55}4_{k-1} + \frac{1}{55}11^{k-1} + \frac{1}{40}(-5)^{k-1} & \text{if } z \geq 1 \\ \end{cases} \]  

(62)

Now, it might look like this is a very different answer, but it isn’t, expression (59) and expression (62) are actually the same. Now that putting \( k = 1 \) in (62) gives

\[ -\frac{2}{55}0 + \frac{1}{55}11^0 + \frac{1}{40}(-5)^0 = -\frac{2}{55} + \frac{1}{88} + \frac{1}{40} = 0 \]  

(63)

and, what’s more, \( 11^{k-1} = 11 \times 11^{k-2} \) and \( (-5)^{k-1} = -5 \times (-5)^{k-2} \).
2 Exercises

1. Solve the difference equation \( x_{k+2} - 4x_{k+1} - 5x_k = 0 \) with \( x_0 = 0 \) and \( x_1 = 1 \).

2. Solve the difference equation \( x_{k+2} - 9x_{k+1} + 20x_k = 2^k \) with \( x_0 = 0 \) and \( x_1 = 0 \).

3. Solve the difference equation \( x_{k+2} + 5x_{k+1} + 6x_k = (-2)^k \) with \( x_0 = 0 \) and \( x_1 = 0 \).

4. Solve the difference equation \( x_{k+2} + 2x_{k+1} - 48x_k = 0 \) with \( x_0 = 4 \) and \( x_1 = 2 \).

5. Solve the difference equation \( x_{k+2} + 7x_{k+1} - 18x_k = \delta_k \) with \( x_0 = 0 \) and \( x_1 = 0 \).

1. So take the Z-transform of both sides

\[
z^2X - z - 4zX - 5X = 0
\]

and move things around to get \( X/z \) on one side and then do partial fractions

\[
\frac{1}{z}X = \frac{1}{z^2 - 4z - 5} = \frac{1}{(z - 5)(z + 1)} = \frac{A}{z - 5} + \frac{B}{z + 1}
\]

In the usual way, we have

\[
1 = A(z + 1) + B(z - 5)
\]

and putting \( z = 5 \) gives \( A = 1/6 \) and putting \( z = -1 \) gives \( B = -1/6 \). Now

\[
X = \frac{z}{6(z - 5)} - \frac{1}{6(z + 1)}
\]

and hence

\[
x_k = \frac{1}{6}z^k - \frac{1}{6}(-1)^k
\]

2. So, in this example, the right hand side of the difference equation is not zero. Taking the Z-transform of both sides we get

\[
z^2X - 9zX + 20X = z[2^k] = \frac{z}{z - 2}
\]

Hence, since \( z^2 - 9z + 20 = (z - 5)(z - 4) \)

\[
\frac{1}{z}X = \frac{1}{(z - 5)(z - 4)(z - 2)}
\]

The usual partial fractions tells us that

\[
\frac{1}{(z - 5)(z - 4)(z - 2)} = \frac{1}{3(z - 5)} - \frac{1}{2(z - 4)} + \frac{1}{6(z - 2)}
\]

and so

\[
x_k = \frac{1}{3}2^k - \frac{1}{2}4^k + \frac{1}{6}2^k
\]

3. Again, taking the Z-transport of both sides we have

\[
z^2X + 5zX + 6X = \frac{z}{z + 2}
\]

Now, since \( z^2 + 5z + 6 = (z + 2)(z + 3) \)

\[
\frac{1}{z}X = \frac{1}{(z + 2)(z + 3)}
\]
and there is a repeated root. The partial fraction expansion with a repeated root includes the root and its square, so we get
\[ \frac{1}{(z + 2)^2(z + 3)} = \frac{A}{z + 2} + \frac{B}{(z + 2)^2} + \frac{C}{z + 3} \] (75)
and so
\[ 1 = A(z + 2)(z + 3) + B(z + 3) + C(z + 2)^2 \] (76)
Choosing \( z = -2 \) gives \( B = 1 \) and \( z = -3 \) gives \( C = 1 \). No value of \( z \) will give \( A \) on its own, so we choose another convenient value and put in the known values of \( B \) and \( C \):
\[ 1 = 6A + 3 + 4 \] (77)
so \( A = -1 \). Now, this means
\[ X = -\frac{z}{z + 2} + \frac{z}{(z + 2)^2} + \frac{z}{z + 3} \] (78)
and so
\[ x_k = (-2)^k + k(-2)^{k-1} + (-3)^k \] (79)
4. Take the Z-transform of both sides, taking care to note the initial conditions
\[ z^2X - 4z - 2z + 2(zX - 4z) - 48X = 0 \] (80)
Thus
\[ z^2X + 2zX - 48X = 4z^2 - 10z \] (81)
giving
\[ \frac{1}{z}X = \frac{4z - 10}{(z + 8)(z - 6)} = \frac{A}{z + 8} + \frac{B}{z - 6} \] (82)
Multiplying across we get
\[ 4z - 10 = A(z - 6) + B(z + 8) \] (83)
Choosing \( z = -8 \) we have
\[ -42 = -14A \] (84)
implying \( A = 3 \). Choosing \( z = 6 \)
\[ 14 = 14B \] (85)
so \( B = 1 \) and we get
\[ X = \frac{3z}{z + 8} + \frac{z}{z - 6} \] (86)
and
\[ x_k = 3(-8)^k + 6^k \] (87)
5. Now, taking the Z-transform and using \( Z[(\delta_k)] = 1 \)
\[ z^2X + 7zX - 18X = 1 \] (88)
and so
\[ X = \frac{1}{z^2 + 7z - 18} = \frac{1}{(z - 9)(z + 2)} = \frac{1}{11(z - 9)} - \frac{1}{11(z + 2)} \] (89)
Thus
\[ X = \frac{1}{z} \left( \frac{z}{11(z - 9)} - \frac{z}{11(z + 2)} \right) \] (90)
and so, using the delay theorem, we have
\[ x_k = \begin{cases} 0 & k = 0 \\ \frac{9^k - 1}{11} - \frac{(-2)^{k-1}}{11} & k > 0 \end{cases} \] (91)