Notes on the Z-transform, part 4[1]

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1 Solving difference equations

At the end of note 3 we saw how to solve a difference equation using Z-transforms. Here is a similar example, solve

$$x_{k+2} - 6x_{k+1} - 55x_k = 0 \tag{1}$$

with $x_1 = 1$ and $x_0 = 0$. We begin by taking the Z-tranform of both sides, remember, if we write $\mathcal{Z}[(x_k)] = X(z)$ then

$$\begin{aligned} \mathcal{Z}[(x_{k+1})] &= zX - zx_0 \\ \mathcal{Z}[(x_{k+2})] &= z^2X - z^2x_0 - zx_1 \end{aligned} \tag{2}$$

so, in this case we get

$$z^X - z - 6zX - 55X = 0 \tag{3}$$

or

$$X = \frac{z}{z^2 - 6z - 55} \tag{4}$$

As before, we do a partial fraction expansion, but, first we move the z over to the right hand side,

$$\frac{1}{z}X = \frac{1}{z^2 - 6z - 55} = \frac{1}{(z-11)(z+5)} = \frac{A}{z-11} + \frac{B}{z+5} \tag{5}$$

giving

$$1 = A(z+5) + B(z-11) \tag{6}$$

Choose $z = 11$ to learn $A = 1/16$ and $z = -5$ to learn $B = -16$. Therefore,

$$\frac{1}{z}X = -\frac{1}{16(z-11)} + \frac{1}{16(z+5)} \tag{7}$$

or

$$X = -\frac{z}{16(z-11)} + \frac{z}{16(z+5)} \tag{8}$$

Both terms of the form $z/(z-r)$ and so

$$x_k = -\frac{1}{16}(11)^k + \frac{1}{16}(-5)^k \tag{9}$$

The basic process is simple, you take the Z-tranform of both sides, you solve for X, you use partial fractions to put it into a convenient form and then work out x_k. In the rest of this note we will look at a variety of examples which exhibit the various difficulties that might be encountered doing this.

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1.1 Not zero on the right hand side

Consider the difference equation

$$x_{k+2} - 6x_{k+1} - 55x_k = -(-3)^k \tag{10}$$

with $x_1 = 0$ and $x_0 = 0$. This is different than the previous example in that the right hand side is not zero, it is 3^k. This doesn't make such a difference, take the Z-tranform of both sides, and noting the trivial initial data ($x_0 = 0$ and $x_1 = 0$),

$$z^2X - 6zX - 55X = \mathcal{Z}[(3^k)] = -\frac{z}{z+3} \tag{11}$$

thus,

$$\frac{1}{z}X = -\frac{1}{(z-11)(z+5)(z+3)} \tag{12}$$

Now, use partial fractions

$$\frac{1}{(z-11)(z+5)(z+3)} = \frac{A}{z-11} + \frac{B}{z+5} + \frac{C}{z+3} \tag{13}$$

or

$$1 = A(z+5)(z+3) + B(z-11)(z+3) + C(z-11)(z+5) \tag{14}$$

Now, the usual, $z = 11$ gives $A = 1/224$, $z = -5$ gives $B = 1/32$ and $z = -3$ gives $C = -1/28$. Not forgetting the minus in equation (12) this gives

$$X = -\frac{1}{224}\frac{z}{z-11} - \frac{1}{32}\frac{z}{z+5} + \frac{1}{28}\frac{z}{z+3} \tag{15}$$

and so

$$x_k = -\frac{1}{224}11^k - \frac{1}{32}(-5)^k + \frac{1}{28}(-3)^k \tag{16}$$

Remember that the sequence on the right hand side might be a sequence of ones, $1 = 1^k$, so, consider

$$x_{k+2} - 6x_{k+1} - 55x_k = 1 \tag{17}$$

with $x_1 = 0$ and $x_0 = 0$. Take the z-tranform of both sides

$$X - 6zX - 55X = \mathcal{Z}[(1)] = \mathcal{Z}[(1^k)] = \frac{z}{z-1} \tag{18}$$

and so

$$\frac{1}{z}X = \frac{1}{(z-11)(z+5)(z-1)} \tag{19}$$

Without going through the calculation, the partial fraction expansion is

$$\frac{1}{(z-11)(z+5)(z-1)} = \frac{1}{160}\frac{1}{z-11} + \frac{1}{96}\frac{1}{z+5} - \frac{1}{60}\frac{1}{z-1} \tag{20}$$

and so,

$$x_k = \frac{1}{160}11^k + \frac{1}{96}(-5)^k - \frac{1}{60} \tag{21}$$

1.2 Repeated root

As happens with Laplace transforms, there can be a repeated root. Consider

$$x_{k+2} - 6x_{k+1} - 55x_k = 11^k \tag{22}$$

with $x_1 = 0$ and $x_0 = 0$. Take the z-tranform of both sides

$$X - 6zX - 55X == \mathcal{Z}[(11^k)] = \frac{z}{z-11} \tag{23}$$

and so

$$\frac{1}{z}X = \frac{1}{(z-11)^2(z+5)} \tag{24}$$

Now, remember that for repeated root the partial fraction expansion has a term with the root and a term with its square:

$$\frac{1}{(z-11)^2(z+5)} = \frac{A}{z-11} + \frac{B}{(z-11)^2} + \frac{C}{z+5} \tag{25}$$

Thus,

$$1 = A(z-11)(z-5) + B(z+5) + C(z-11)^2 \tag{26}$$

Now, $z = 11$ gives $B = 1/16$ and $z = -5$ gives $C = 1/256$. The problem is that no choice of z gives A on its own, instead we chose any value that hasn't been used before, $z = 0$ for example,

$$1 = -55A + 5B + 121C \tag{27}$$

and now, we substitute for the known values of B and C,

$$1 = -55A + \frac{5}{16} + \frac{121}{256} = -55A + \frac{201}{256} \tag{28}$$

Hence

$$-55A = 1 - \frac{201}{256} = \frac{55}{256} \tag{29}$$

so $A = -1/256$. This means that

$$X = -\frac{1}{256}\frac{z}{z-11} + \frac{1}{16}\frac{z}{(z-11)^2} + \frac{1}{256}\frac{z}{z+5} \tag{30}$$

The only problem now is that the $z/(z-11)^2$ term might look unfamiliar, but recall

$$\mathcal{Z}[(kr^{k-1})] = \frac{z}{(z-r)^2} \tag{31}$$

has this form. We get

$$x_k = -\frac{1}{256}11^k + \frac{k}{16}11^{k-1} + \frac{1}{256}(-5)^k \tag{32}$$

1.3 Less convenient inital data

So far the values of x_0 and x_1 have been chosen to keep things as simple as possible. More general values of x_0 and x_1 might be less convenient, but there is no big change in the method. Consider

$$x_{k+2} - 6x_{k+1} - 55x_k = 0 \tag{33}$$

with $x_1 = 2$ and $x_0 = 6$. Taking the Z-tranform of both sides gives

$$zX^2 - z^2x_0 - zx_1 - 6(zX - zx_0) - 55X = 0 \tag{34}$$

and substituting for the initial data

$$zX^2 - 6z^2 - 2z - 6zX + 36z - 55X = 0 \tag{35}$$

Moving everything around, this gives,

$$X = \frac{6z^2 - 34z}{z^2 - 6z - 55} \tag{36}$$

We still want a z on top when we are finished, so move one over:

$$\frac{1}{z}X = \frac{6z - 34}{z^2 - 6z - 55} \tag{37}$$

and now, remember that the partial fraction expansion works fine provided whats on top is a polynomial of degree less than than the polynomial on the bottom, so we have

$$\frac{6z - 34}{(z-11)(z+5)} = \frac{A}{z-11} + \frac{B}{z+5} \tag{38}$$

or

$$6z - 34 = A(z+5) + B(z-11) \tag{39}$$

Choose $z = 11$ to get

$$66 - 34 = 16A \tag{40}$$

or $A = 2$. $z = -5$ gives

$$-30 - 34 = -16B \tag{41}$$

or $B = 4$. Thus,

$$X = \frac{2}{z-11} + \frac{4}{z+5} \tag{42}$$

and

$$x_k = 2(11)^k + 4(-5)^k \tag{43}$$

1.4 Examples involving the delay theorem

Consider

$$x_{k+2} - 6x_{k+1} - 55x_k = \delta_k \tag{44}$$

with $x_1 = 0$ and $x_0 = 0$. δ_k is the unit pulse

$$(\delta_k) = (1, 0, 0, 0, \ldots) \tag{45}$$

and $\mathcal{Z}[(\delta_k)] = 1$. Hence

$$z^2X - 6zX - 55X = \mathcal{Z}[(\delta_k)] = 1 \tag{46}$$

so

$$X = \frac{1}{(z-11)(z+5)} \tag{47}$$

Using the partial fraction expansion, this gives

$$X = \frac{1}{16}\frac{1}{z-11} - \frac{1}{16}\frac{1}{z+5} \tag{48}$$

The problem now is that there are no z's on top. However, if we rewrite it as

$$X = \frac{1}{z}\left[\frac{1}{16}\frac{z}{z-11} - \frac{1}{16}\frac{z}{z+5}\right] \tag{49}$$

Now, the part inside the square brackets has the form we are familiar with, we can see

$$\mathcal{Z}\left[\left(\frac{1}{16}11^k - \frac{1}{16}(-5)^k\right)\right] = \frac{1}{16}\frac{z}{z-11} - \frac{1}{16}\frac{z}{z+5} \tag{50}$$

and we also know from the delay theorem that the affect of multiplying by $1/z^{k_0}$ is to delay the sequence by k_0 steps. Hence, the sequence here is delayed by one step and

$$x_k = \begin{cases} 0 & z = 0 \\ \frac{1}{16}11^{k-1} - \frac{1}{16}(-5)^{k-1} & z \geq 1 \end{cases} \tag{51}$$

Of course, the sequence on the right might be more complicated, consider

$$x_{k+2} - 6x_{k+1} - 55x_k = y_k \tag{52}$$

with $x_1 = 0$ and $x_0 = 0$ where

$$(y_k) = (0, 2, 0, 0, 0, \ldots) \tag{53}$$

We have to calculate $\mathcal{Z}[(y_k)]$ before we can make any progress. However, it is easy to see that (y_k) is the first delay of twice the unit pulse $(y_k) = 2(\delta_{k-1})$ so

$$\mathcal{Z}[(y_k)] = \frac{2}{z} \tag{54}$$

Thus the Z-tranform of the difference equation gives

$$z^X - 6zX - 55X = \frac{2}{z} \tag{55}$$

so,

$$X = \frac{2}{z(z-11)(z+5)} \tag{56}$$

There are two ways to go on from here, the first is to use the previous partial fractions expansion

$$\begin{aligned} X &= \frac{2}{z}\left[\frac{1}{16}\frac{1}{z-11} - \frac{1}{16}\frac{1}{z+5}\right] && (57) \\ &= \frac{2}{z^2}\left[\frac{1}{16}\frac{z}{z-11} - \frac{1}{16}\frac{z}{z+5}\right] && (58) \end{aligned}$$

so now we are dealing with a two step delay and, keeping the extra factor of two in mind

$$x_k = \begin{cases} 0 & z \leq 1 \\ \frac{1}{8}11^{k-2} - \frac{1}{8}(-5)^{k-2} & z \geq 2 \end{cases} \tag{59}$$

The other way to make progress is to work out the partial fraction expansion

$$\frac{1}{z(z-11)(z+5)} = -\frac{1}{55}\frac{1}{z} + \frac{1}{176}\frac{1}{z-11} + \frac{1}{80}\frac{1}{z+5} \tag{60}$$

This means that

$$X = \frac{2}{z}\left[-\frac{1}{55} + \frac{1}{176}\frac{z}{z-11} + \frac{1}{80}\frac{z}{z+5}\right] \tag{61}$$

and using the delay theorem

$$x_k = \begin{cases} 0 & z = 0 \\ -\frac{2}{55}\delta_{k-1} + \frac{1}{88}(11^{k-1} + \frac{1}{40}(-5)^{k-1} & z \geq 1 \end{cases} \tag{62}$$

Now, it might look like this is a very different answer, but it isn't, expression (59) and expression (62) are actually the same. Now that putting $k = 1$ in (62) gives

$$-\frac{2}{55}\delta_0 + \frac{1}{88}11^0 + \frac{1}{40}(-5)^0 = -\frac{2}{55} + \frac{1}{88} + \frac{1}{40} = 0 \tag{63}$$

and, what's more, $11^{k-1} = 11 \times 11^{k-2}$ and $(-5)^{k-1} = -5 \times (-5)^{k-2}$.

2 Exercises

1. Solve the difference equation $x_{k+2} - 4x_{k+1} - 5x_k = 0$ with $x_0 = 0$ and $x_1 = 1$.
2. Solve the difference equation $x_{k+2} - 9x_{k+1} + 20x_k = 2^k$ with $x_0 = 0$ and $x_1 = 0$.
3. Solve the difference equation $x_{k+2} + 5x_{k+1} + 6x_k = (-2)^k$ with $x_0 = 0$ and $x_1 = 0$.
4. Solve the difference equation $x_{k+2} + 2x_{k+1} - 48x_k = 0$ with $x_0 = 4$ and $x_1 = 2$.
5. Solve the difference equation $x_{k+2} + 7x_{k+1} - 18x_k = \delta_k$ with $x_0 = 0$ and $x_1 = 0$.

1. So take the Z-transform of both sides

$$z^2X - z - 4zX - 5X = 0 \tag{64}$$

and move things around to get X/z on one side and then do partial fractions

$$\frac{1}{z}X = \frac{1}{z^2 - 4z - 5} = \frac{1}{(z-5)(z+1)} = \frac{A}{z-5} + \frac{B}{z+1} \tag{65}$$

In the usual way, we have

$$1 = A(z+1) + B(z-5) \tag{66}$$

and putting $z = 5$ gives $A = 1/6$ and putting $z = -1$ gives $B = -1/6$. Now

$$X = \frac{z}{6(z-5)} - \frac{1}{6(z+1)} \tag{67}$$

and hence

$$x_k = \frac{1}{6}5^k - \frac{1}{6}(-1)^k \tag{68}$$

2. So, in this example, the right hand side of the difference equation is not zero. Taking the Z-transform of both sides we get

$$z^2X - 9zX + 20X = \mathcal{Z}[(2^k)] = \frac{z}{z-2} \tag{69}$$

Hence, since $z^2 - 9z + 20 = (z-5)(z-4)$

$$\frac{1}{z}X = \frac{1}{(z-5)(z-4)(z-2)} \tag{70}$$

The usual partial fractions tells us that

$$\frac{1}{(z-5)(z-4)(z-2)} = \frac{1}{3(z-5)} - \frac{1}{2(z-4)} + \frac{1}{6(z-2)} \tag{71}$$

and so

$$x_k = \frac{1}{3}5^k - \frac{1}{2}4^k + \frac{1}{6}2^k \tag{72}$$

3. Again, taking the Z-tranform of both sides we have

$$z^2X + 5zX + 6X = \frac{z}{z+2} \tag{73}$$

Now, since $z^2 + 5z + 6 = (z+2)(z+3)$

$$\frac{1}{z}X = \frac{1}{(z+2)^2(z+3)} \tag{74}$$

and there is a repeated root. The partial fraction expansion with a repeated root includes the root and its square, so we get

$$\frac{1}{(z+2)^2(z+3)} = \frac{A}{z+2} + \frac{B}{(z+2)^2} + \frac{C}{z+3} \tag{75}$$

and so

$$1 = A(z+2)(z+3) + B(z+3) + C(z+2)^2 \tag{76}$$

Choosing $z = -2$ gives $B = 1$ and $z = -3$ gives $C = 1$. No value of z will give A on its own, so we choose another convenient value and put in the known values of B and C:

$$1 = 6A + 3 + 4 \tag{77}$$

so $A = -1$. Now, this means

$$X = -\frac{z}{z+2} + \frac{z}{(z+2)^2} + \frac{z}{z+3} \tag{78}$$

and so

$$x_k = (-2)^k + k(-2)^{k-1} + (-3)^k \tag{79}$$

4. Take the Z-tranform of both sides, taking care to note the initial conditions

$$z^2X - 4z^2 - 2z + 2(zX - 4z) - 48X = 0 \tag{80}$$

Thus

$$z^2X + 2zX - 48X = 4z^2 - 10z \tag{81}$$

giving

$$\frac{1}{z}X = \frac{4z-10}{(z+8)(z-6)} = \frac{A}{z+8} + \frac{B}{z-6} \tag{82}$$

Multiplying across we get

$$4z - 10 = A(z-6) + B(z+8) \tag{83}$$

Choosing $z = -8$ wehave

$$-42 = -14A \tag{84}$$

implying $A = 3$. Choosing $z = 6$

$$14 = 14B \tag{85}$$

so $B = 1$ and we get

$$X = \frac{3z}{z+8} + \frac{z}{z-6} \tag{86}$$

and

$$x_k = 3(-8)^k + 6^k \tag{87}$$

5. Now, taking the Z-transform and using $\mathcal{Z}[(\delta_k)] = 1$

$$z^2X + 7zX - 18X = 1 \tag{88}$$

and so

$$X = \frac{1}{z^2+7z-18} = \frac{1}{(z-9)(z+2)} = \frac{1}{11(z-9)} - \frac{1}{11(z+2)} \tag{89}$$

Thus

$$X = \frac{1}{z}\left(\frac{z}{11(z-9)} - \frac{z}{11(z+2)}\right) \tag{90}$$

and so, using the delay theorem, we have

$$x_k = \begin{cases} 0 & k = 0 \\ \frac{1}{11}9^{k-1} - \frac{1}{11}(-2)^{k-1} & k > 0 \end{cases} \tag{91}$$