Notes on the Z-transform, part 4

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1 Solving difference equations

At the end of note 3 we saw how to solve a difference equation using Z-transforms. Here is a similar example, solve

\[ x_{k+2} - 6x_{k+1} - 55x_k = 0 \]  

with \( x_1 = 1 \) and \( x_0 = 0 \). We begin by taking the Z-transform of both sides, remember, if we write \( Z[(x_k)] = X(z) \) then

\[
\begin{align*}
Z[(x_{k+1})] &= zX - zx_0 \\
Z[(x_{k+2})] &= z^2X - z^2x_0 - zx_1
\end{align*}
\]  

so, in this case we get

\[ zX - z - 6zX - 55X = 0 \]  

or

\[ X = \frac{z}{z^2 - 6z - 55} \]  

As before, we do a partial fraction expansion, but, first we move the \( z \) over to the right hand side,

\[
\frac{1}{z}X = \frac{1}{z^2 - 6z - 55} = \frac{1}{(z - 11)(z + 5)} = \frac{A}{z - 11} + \frac{B}{z + 5}
\]

\[ X = \frac{z}{z^2 - 6z - 55} \]

\[ \frac{1}{z}X = \frac{1}{16(z - 11)} + \frac{1}{16(z + 5)} \]

or

\[ X = -\frac{z}{16(z - 11)} + \frac{z}{16(z + 5)} \]

Both terms of the form \( z/(z - r) \) and so

\[ x_k = -\frac{1}{16}(11)^k + \frac{1}{16}(-5)^k \]

The basic process is simple, you take the Z-tranform of both sides, you solve for \( X \), you use partial fractions to put it into a convenient form and then work out \( x_k \). In the rest of this note we will look at a variety of examples which exhibit the various difficulties that might be encountered doing this.

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1.1 Not zero on the right hand side

Consider the difference equation

\[ x_{k+2} - 6x_{k+1} - 55x_k = -(-3)^k \] (10)

with \( x_1 = 0 \) and \( x_0 = 0 \). This is different than the previous example in that the right hand side is not zero, it is \( 3^k \). This doesn’t make such a difference, take the Z-transform of both sides, and noting the trivial initial data \( (x_0 = 0 \text{ and } x_1 = 0) \),

\[ z^2 X - 6zX - 55X = Z[(3^k)] = -\frac{z}{z + 3} \] (11)

thus,

\[ \frac{1}{z} X = -\frac{1}{(z - 11)(z + 5)(z + 3)} \] (12)

Now, use partial fractions

\[ \frac{1}{(z - 11)(z + 5)(z + 3)} = \frac{A}{z - 11} + \frac{B}{z + 5} + \frac{C}{z + 3} \] (13)

or

\[ 1 = A(z + 5)(z + 3) + B(z - 11)(z + 3) + C(z - 11)(z + 5) \] (14)

Now, the usual, \( z = 11 \) gives \( A = 1/224 \), \( z = -5 \) gives \( B = 1/32 \) and \( z = -3 \) gives \( C = -1/28 \). Not forgetting the minus in equation (12) this gives

\[ X = -\frac{1}{224} \frac{z}{z - 11} - \frac{1}{32} \frac{z}{z + 5} + \frac{1}{28} \frac{z}{z + 3} \] (15)

and so

\[ x_k = -\frac{1}{224} 11^k - \frac{1}{32} (-5)^k + \frac{1}{28} (-3)^k \] (16)

Remember that the sequence on the right hand side might be a sequence of ones, \( 1 = 1^k \), so, consider

\[ x_{k+2} - 6x_{k+1} - 55x_k = 1 \] (17)

with \( x_1 = 0 \) and \( x_0 = 0 \). Take the z-transform of both sides

\[ X - 6zX - 55X = Z[(1)] = Z[(1^k)] = \frac{z}{z - 1} \] (18)

and so

\[ \frac{1}{z} X = \frac{1}{(z - 11)(z + 5)(z - 1)} \] (19)

Without going through the calculation, the partial fraction expansion is

\[ \frac{1}{(z - 11)(z + 5)(z - 1)} = \frac{1}{160} \frac{1}{z - 11} + \frac{1}{96} \frac{1}{z + 5} - \frac{1}{60} \frac{1}{z - 1} \] (20)

and so,

\[ x_k = \frac{1}{160} 11^k + \frac{1}{96} (-5)^k - \frac{1}{60} \] (21)
1.2 Repeated root

As happens with Laplace transforms, there can be a repeated root. Consider

\[ x_{k+2} - 6x_{k+1} - 55x_k = 11^k \tag{22} \]

with \( x_1 = 0 \) and \( x_0 = 0 \). Take the \( z \)-transform of both sides

\[ X - 6zX - 55X = Z[(11^k)] = \frac{z}{z - 11} \tag{23} \]

and so

\[ \frac{1}{z}X = \frac{1}{(z - 11)^2(z + 5)} \tag{24} \]

Now, remember that for repeated root the partial fraction expansion has a term with the root and a term with its square:

\[ \frac{1}{(z - 11)^2(z + 5)} = \frac{A}{z - 11} + \frac{B}{z - 11} + \frac{C}{z + 5} \tag{25} \]

Thus,

\[ 1 = A(z - 11)(z - 5) + B(z + 5) + C(z - 11)^2 \tag{26} \]

Now, \( z = 11 \) gives \( B = \frac{1}{16} \) and \( z = -5 \) gives \( C = \frac{1}{256} \). The problem is that no choice of \( z \) gives \( A \) on its own, instead we chose any value that hasn’t been used before, \( z = 0 \) for example,

\[ 1 = -55A + 5B + 121C \tag{27} \]

and now, we substitute for the known values of \( B \) and \( C \),

\[ 1 = -55A + \frac{5}{16} + \frac{121}{256} = -55A + \frac{201}{256} \tag{28} \]

Hence

\[ -55A = 1 - \frac{201}{256} = \frac{55}{256} \tag{29} \]

so \( A = -\frac{1}{256} \). This means that

\[ X = -\frac{1}{256}\frac{z}{z - 11} + \frac{1}{16}\frac{z}{(z - 11)^2} + \frac{1}{256}\frac{z}{z + 5} \tag{30} \]

The only problem now is that the \( z/(z - 11)^2 \) term might look unfamiliar, but recall

\[ Z[(kr^{k-1})] = \frac{z}{(z - r)^2} \tag{31} \]

has this form. We get

\[ x_k = -\frac{1}{256}11^k + \frac{k}{16}11^{k-1} + \frac{1}{256}(-5)^k \tag{32} \]
1.3 Less convenient initial data

So far the values of \( x_0 \) and \( x_1 \) have been chosen to keep things as simple as possible. More general values of \( x_0 \) and \( x_1 \) might be less convenient, but there is no big change in the method. Consider

\[
x_{k+2} - 6x_{k+1} - 55x_k = 0
\]

with \( x_1 = 2 \) and \( x_0 = 6 \). Taking the Z-transform of both sides gives

\[
zX^2 - z^2x_0 - zx_1 - 6(zX - zx_0) - 55X = 0
\]

and substituting for the initial data

\[
zX^2 - 6z^2 - 2z - 6zX + 36z - 55X = 0
\]

Moving everything around, this gives,

\[
X = \frac{6z^2 - 34z}{z^2 - 6z - 55}
\]

We still want a \( z \) on top when we are finished, so move one over:

\[
\frac{1}{z}X = \frac{6z - 34}{z^2 - 6z - 55}
\]

and now, remember that the partial fraction expansion works fine provided what’s on top is a polynomial of degree less than than the polynomial on the bottom, so we have

\[
\frac{6z - 34}{(z - 11)(z + 5)} = \frac{A}{z - 11} + \frac{B}{z + 5}
\]

or

\[
6z - 34 = A(z + 5) + B(z - 11)
\]

Choose \( z = 11 \) to get

\[
66 - 34 = 16A
\]

or \( A = 2 \). \( z = -5 \) gives

\[
-30 - 34 = -16B
\]

or \( B = 4 \). Thus,

\[
X = \frac{2}{z - 11} + \frac{4}{z + 5}
\]

and

\[
x_k = 2(11)^k + 4(-5)^k
\]
1.4 Examples involving the delay theorem

Consider
\[ x_{k+2} - 6x_{k+1} - 55x_k = \delta_k \] (44)
with \( x_1 = 0 \) and \( x_0 = 0 \). \( \delta_k \) is the unit pulse
\[ (\delta_k) = (1, 0, 0, 0, \ldots) \] (45)
and \( Z[(\delta_k)] = 1 \). Hence
\[ z^2X - 6zX - 55X = Z[(\delta_k)] = 1 \] (46)
so
\[ X = \frac{1}{(z - 11)(z + 5)} \] (47)
Using the partial fraction expansion, this gives
\[ X = \frac{1}{16} \frac{1}{z - 11} - \frac{1}{16} \frac{1}{z + 5} \] (48)
The problem now is that there are no \( z \)'s on top. However, if we rewrite it as
\[ X = \frac{1}{z} \left[ \frac{1}{16} \frac{z}{z - 11} - \frac{1}{16} \frac{z}{z + 5} \right] \] (49)
Now, the part inside the square brackets has the form we are familiar with, we can see
\[ Z \left[ \left( \frac{1}{16} \frac{11^k}{16} - \frac{1}{16} (-5)^k \right) \right] = \frac{1}{16} \frac{z}{z - 11} - \frac{1}{16} \frac{z}{z + 5} \] (50)
and we also know from the delay theorem that the affect of multiplying by \( 1/z^{k_0} \) is to delay the sequence by \( k_0 \) steps. Hence, the sequence here is delayed by one step and
\[ x_k = \begin{cases} 0 & z = 0 \\ \frac{1}{16} \frac{11^{k-1}}{16} - \frac{1}{16} (-5)^{k-1} & z \geq 1 \end{cases} \] (51)
Of course, the sequence on the right might be more complicated, consider
\[ x_{k+2} - 6x_{k+1} - 55x_k = y_k \] (52)
with \( x_1 = 0 \) and \( x_0 = 0 \) where
\[ (y_k) = (0, 2, 0, 0, 0, \ldots) \] (53)
We have to calculate \( Z[(y_k)] \) before we can make any progress. However, it is easy to see that \( (y_k) \) is the first delay of twice the unit pulse \( (y_k) = 2(\delta_{k-1}) \) so
\[ Z[(y_k)] = \frac{2}{z} \] (54)
Thus the Z-transform of the difference equation gives

$$z^{X} - 6zhX - 55X = \frac{2}{z}$$  \hspace{1cm} (55)

so,

$$X = \frac{2}{z(z - 11)(z + 5)}$$  \hspace{1cm} (56)

There are two ways to go on from here, the first is to use the previous partial fractions expansion

$$X = \frac{2}{z} \left[ \frac{1}{16z - 11} - \frac{1}{16z + 5} \right]$$  \hspace{1cm} (57)

$$= \frac{2}{z^2} \left[ \frac{1}{16z - 11} - \frac{1}{16z + 5} \right]$$  \hspace{1cm} (58)

so now we are dealing with a two step delay and, keeping the extra factor of two in mind

$$x_k = \begin{cases} 
0 & z \leq 1 \\
\frac{1}{8}11^{k-2} - \frac{1}{8}(-5)^{k-2} & z \geq 2
\end{cases}$$  \hspace{1cm} (59)

The other way to make progress is to work out the partial fraction expansion

$$\frac{1}{z(z - 11)(z + 5)} = -\frac{1}{55z} + \frac{1}{176z - 11} + \frac{1}{80z + 5}$$  \hspace{1cm} (60)

This means that

$$X = \frac{2}{z} \left[ -\frac{1}{55} + \frac{1}{176} \frac{z}{z - 11} + \frac{1}{80} \frac{z}{z + 5} \right]$$  \hspace{1cm} (61)

and using the delay theorem

$$x_k = \begin{cases} 
0 & z = 0 \\
-\frac{2}{55}\delta_{k-1} + \frac{1}{88}(11^{k-1} + \frac{1}{40}(-5)^{k-1}) & z \geq 1
\end{cases}$$  \hspace{1cm} (62)

Now, it might look like this is a very different answer, but it isn't, expression (59) and expression (62) are actually the same. Now that putting $k = 1$ in (62) gives

$$-\frac{2}{55}\delta_0 + \frac{1}{88}11^0 + \frac{1}{40}(-5)^0 = -\frac{2}{55} + \frac{1}{88} + \frac{1}{40} = 0$$  \hspace{1cm} (63)

and, what's more, $11^{k-1} = 11 \times 11^{k-2}$ and $(-5)^{k-1} = -5 \times (-5)^{k-2}$.
2 Exercises

1. Solve the difference equation \( x_{k+2} - 4x_{k+1} - 5x_k = 0 \) with \( x_0 = 0 \) and \( x_1 = 1 \).

2. Solve the difference equation \( x_{k+2} - 9x_{k+1} + 20x_k = 2^k \) with \( x_0 = 0 \) and \( x_1 = 0 \).

3. Solve the difference equation \( x_{k+2} + 5x_{k+1} + 6x_k = (-2)^k \) with \( x_0 = 0 \) and \( x_1 = 0 \).

4. Solve the difference equation \( x_{k+2} + 2x_{k+1} - 48x_k = 0 \) with \( x_0 = 4 \) and \( x_1 = 2 \).

5. Solve the difference equation \( x_{k+2} + 7x_{k+1} - 18x_k = \delta_k \) with \( x_0 = 0 \) and \( x_1 = 0 \).
1. So take the Z-transform of both sides

\[ z^2X - z - 4zX - 5X = 0 \]  

and move things around to get \( X/z \) on one side and then do partial fractions

\[ \frac{1}{z}X = \frac{1}{z^2 - 4z - 5} = \frac{1}{(z - 5)(z + 1)} = \frac{A}{z - 5} + \frac{B}{z + 1} \]  

In the usual way, we have

\[ 1 = A(z + 1) + B(z - 5) \]  

and putting \( z = 5 \) gives \( A = 1/6 \) and putting \( z = -1 \) gives \( B = -1/6 \). Now

\[ X = \frac{z}{6(z - 5)} - \frac{1}{6(z + 1)} \]  

and hence

\[ x_k = \frac{1}{6}5^k - \frac{1}{6}(-1)^k \]  

2. So, in this example, the right hand side of the difference equation is not zero. Taking the Z-transform of both sides we get

\[ z^2X - 9zX + 20X = Z[(2^k)] = \frac{z}{z - 2} \]  

Hence, since \( z^2 - 9z + 20 = (z - 5)(z - 4) \)

\[ \frac{1}{z}X = \frac{1}{(z - 5)(z - 4)(z - 2)} \]  

The usual partial fractions tells us that

\[ \frac{1}{(z - 5)(z - 4)(z - 2)} = \frac{1}{3(z - 5)} - \frac{1}{2(z - 4)} + \frac{1}{6(z - 2)} \]  

and so

\[ x_k = \frac{1}{3}5^k - \frac{1}{2}4^k + \frac{1}{6}2^k \]  

3. Again, taking the Z-transform of both sides we have

\[ z^2X + 5zX + 6X = \frac{z}{z + 2} \]  

Now, since \( z^2 + 5z + 6 = (z + 2)(z + 3) \)

\[ \frac{1}{z}X = \frac{1}{(z + 2)^2(z + 3)} \]
and there is a repeated root. The partial fraction expansion with a repeated root includes the root and its square, so we get

\[
\frac{1}{(z + 2)(z + 3)} = \frac{A}{z + 2} + \frac{B}{(z + 2)^2} + \frac{C}{z + 3}
\]  
(75)

and so

\[
1 = A(z + 2)(z + 3) + B(z + 3) + C(z + 2)^2
\]  
(76)

Choosing \( z = -2 \) gives \( B = 1 \) and \( z = -3 \) gives \( C = 1 \). No value of \( z \) will give \( A \) on its own, so we choose another convenient value and put in the known values of \( B \) and \( C \):

\[
1 = 6A + 3 + 4
\]  
(77)

so \( A = -1 \). Now, this means

\[
X = \frac{z}{z + 2} + \frac{z}{(z + 2)^2} + \frac{z}{z + 3}
\]  
(78)

and so

\[
x_k = (-2)^k + k(-2)^{k-1} + (-3)^k
\]  
(79)

4. Take the Z-tranform of both sides, taking care to note the initial conditions

\[
z^2X - 4z^2 - 2z + 2(zX - 4z) - 48X = 0
\]  
(80)

Thus

\[
z^2X + 2zX - 48X = 4z^2 - 10z
\]  
(81)

giving

\[
\frac{1}{z}X = \frac{4z - 10}{(z + 8)(z - 6)} = \frac{A}{z + 8} + \frac{B}{z - 6}
\]  
(82)

Multiplying across we get

\[
4z - 10 = A(z - 6) + B(z + 8)
\]  
(83)

Choosing \( z = -8 \) we have

\[
-42 = -14A
\]  
(84)

implying \( A = 3 \). Choosing \( z = 6 \)

\[
14 = 14B
\]  
(85)

so \( B = 1 \) and we get

\[
X = \frac{3z}{z + 8} + \frac{z}{z - 6}
\]  
(86)

and

\[
x_k = 3(-8)^k + 6^k
\]  
(87)
5. Now, taking the Z-transform and using $Z[(\delta_k)] = 1$

$$z^2X + 7zX - 18X = 1$$

(88)

and so

$$X = \frac{1}{z^2 + 7z - 18} = \frac{1}{(z - 9)(z + 2)} = \frac{1}{11(z - 9)} - \frac{1}{11(z + 2)}$$

(89)

Thus

$$X = \frac{1}{z} \left( \frac{z}{11(z - 9)} - \frac{z}{11(z + 2)} \right)$$

(90)

and so, using the delay theorem, we have

$$x_k = \begin{cases} 
0 & k = 0 \\
\frac{1}{11}9^{k-1} - \frac{1}{11}(-2)^{k-1} & k > 0 
\end{cases}$$

(91)