Notes on the Z-transform, part 3[1]

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1 The advancing theorem or second shift theorem

The advancing theorem is used for finding the Z-tranform of a sequence which has been advanced. Here is an example, say we have the sequence

$$(x_k) = (3, 6, 12, 24, \ldots) \tag{1}$$

then the first advance of this is the sequence

$$(x_{k+1}) = (6, 12, 24, 48, \ldots) \tag{2}$$

and the second advance is

$$(x_{k+2}) = (12, 24, 48, 96, \ldots) \tag{3}$$

The second shift theorem tells us that

$$\begin{aligned}\mathcal{Z}[(x_{k+1})] &= z\mathcal{Z}[(x_k)] - zx_0 \\ \mathcal{Z}[(x_{k+2})] &= z^2\mathcal{Z}[(x_k)] - z^2x_0 - zx_1\end{aligned} \tag{4}$$

Thus, considering the example with $(x_k) = (3, 6, 12, 24, \ldots)$ above, we have

$$\mathcal{Z}[(x_k)] = \mathcal{Z}[(3, 6, 12, 24, \ldots)] = \mathcal{Z}[3(1, 2, 4, 8, \ldots)] = \frac{3z}{z-2} \tag{5}$$

now,

$$\begin{aligned}\mathcal{Z}[(x_{k+1})] &= z\mathcal{Z}[(x_k)] - zx_0 = \frac{3z^2}{z-2} - 3z \\ &= \frac{3z^2}{z-2} - \frac{3z^2-6z}{z-2} = \frac{6z}{z-2}\end{aligned} \tag{6}$$

and

$$\begin{aligned}\mathcal{Z}[(x_{k+2})] &= z^2\mathcal{Z}[(x_k)] - z^2x_0 - zx_1 = \frac{3z^3}{z-2} - 3z^2 - 6z \\ &= \frac{3z^3}{z-2} - \frac{3z^3-6z^2}{z-2} - \frac{6z^2-12z}{z-2} = \frac{12z}{z-2}\end{aligned} \tag{7}$$

To prove the theorem for the first advance, we go back to first principals and use a change of index:

$$\mathcal{Z}[(x_{k+1})] = \sum_{k=0}^{\infty} \frac{x_{k+1}}{z^k} \tag{8}$$

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now, let $k' = k+1$, so $k = k'-1$ and when $k=0$ we have $k'=1$, when $k=\infty$, then $k'=\infty$ as well. Hence

$$\mathcal{Z}[(x_{k+1})] = \sum_{k=0}^{\infty} \frac{x_{k+1}}{z^k} = \sum_{k'=1}^{\infty} \frac{x_{k'}}{z^{k'-1}} \tag{9}$$

Next we use

$$\frac{1}{z^{k'-1}} = z\frac{1}{z^{k'}} \tag{10}$$

and the z can come to the front of the sum since it has no index:

$$\mathcal{Z}[(x_{k+1})] = z\sum_{k'=1}^{\infty} \frac{x_{k'}}{z^{k'}} \tag{11}$$

Now, the sum starts at one instead of zero, we fix this by adding and subtracting the zeroth term

$$\begin{aligned}
\mathcal{Z}[(x_{k+1})] &= z\sum_{k'=1}^{\infty} \frac{x_{k'}}{z^{k'}} \\
&= z\sum_{k'=1}^{\infty} \frac{x_{k'}}{z^{k'}} + zx_0 - zx_0 \\
&= z\sum_{k'=0}^{\infty} \frac{x_{k'}}{z^{k'}} - zx_0
\end{aligned} \tag{12}$$

Finally, the sum is just $\mathcal{Z}[(x_k)]$, remember, it doesn't matter what we call an index if we are summing over it, $\mathcal{Z}[(x_k)]$ means exactly the same thing as $\mathcal{Z}[(x_{k'})]$, this finishes the proof:

$$\mathcal{Z}[(x_{k+1})] = z\mathcal{Z}[(x_k)] - zx_0 \tag{13}$$

To do the second advance we just apply the first advance formula twice. In short, as well as being the second advance of (x_k), (x_{k+2}) is the first advance of (x_{k+1}). The first term in the sequence (x_{k+1}) is x_1. Applying the formula for the first advance we have

$$\mathcal{Z}[(x_{k+2})] = z\mathcal{Z}[(x_{k+1})] - zx_1 \tag{14}$$

and then applying it again

$$\mathcal{Z}[(x_{k+2})] = z(z\mathcal{Z}[(x_k)] - zx_0) - zx_1 = z^2\mathcal{Z}[(x_k)] - z^2x_0 - zx_1 \tag{15}$$

2 Difference equations

As mentioned before, the main use for Z-tranforms is solving difference equations. An example of a difference equation is

$$x_{k+1} - x_k = 0 \tag{16}$$

with $x_0 = 1$. You see that the equation tells you that the $k + 1$th term in the sequence is equal to the kth term. This doesn't tell you how to start off, but there is also an initial condition given: $x_0 = 1$. So, if $x_0 = 1$, the difference equation says $x_{k+1} = x_k$ and with $k = 0$ this tells us $x_1 = x_0 = 1$, next, with $k = 1$, the difference equation tells us that $x_2 = x_1 = 1$, and so on, clearly every term in this sequence is one and the solution to the difference equation is

$$(x_k) = (1, 1, 1, 1, \ldots) \tag{17}$$

Although this is too easy an example for it to be worthwhile, we could have solved this difference equation by taking the Z-tranform of both sides. Bearing in mind that the Z-transform of the sequence $(0, 0, 0, \ldots)$ is zero, we have,

$$\mathcal{Z}[(x_{k+1}) - (x_k)] = 0 \tag{18}$$

and, hence, if we write $\mathcal{Z}[(x_k)] = X(z)$ we have

$$zX - zx_0 - X = 0 \tag{19}$$

and then, we put in that $x_0 = 1$ and find

$$(z - 1)X = z \tag{20}$$

so

$$X = \frac{z}{z - 1} \tag{21}$$

This is $z/(z - r)$ with $r = 1$ and from the table of Z-tranforms, we see that this means $(x_k) = (1, 1, 1, \ldots)$. This is what we expected.

Here is another difference equation:

$$x_{k+2} - x_{k+1} - 2x_k = 0 \tag{22}$$

with $x_0 = 0$ and $x_1 = 1$. This is called a two-step difference equation because it relates x_{k+2} to the two terms below it in the sequence, x_{k+1} and x_k.

We can work the sequence out term by term. With $k = 0$ we have

$$x_2 - x_1 - 2x_0 = 0 \tag{23}$$

Putting in $x_0 = 0$ and $x_1 = 1$

$$x_2 = 1 \tag{24}$$

Next, with $k = 1$ the difference equation is

$$x_3 - x_2 - 2x_1 = 0 \tag{25}$$

and we know $x_2 = 1$ and $x_1 = 1$, therefore

$$x_3 = 3 \tag{26}$$

and so on it goes, with $k = 2$

$$x_4 - x_3 - 2x_2 = 0 \tag{27}$$

and we know $x_3 = 3$ and $x_2 = 1$, so

$$x_4 = 5 \tag{28}$$

So far we have worked out that the sequence start off $(0, 1, 1, 3, 5, \ldots)$. What we really want is to be able to write x_k in terms of k, we can do this using the Z-tranform.

First, we take the Z-tranform of both sides of the equation

$$\mathcal{Z}[(x_{k+2}) - \mathcal{Z}[(x_{k+1})] - 2\mathcal{Z}[(x_k)] = 0 \tag{29}$$

Using the advancing theorem this means

$$z^2X - z^2x_0 - zx_1 - zX - zx_0 - 2X = 0 \tag{30}$$

Putting in $x_0 = 0$ and $x_1 = 1$

$$z^2X - z - zX - 2X = 0 \tag{31}$$

or

$$X = \frac{z}{z^2 - z - 2} \tag{32}$$

Now that we know $X = \mathcal{Z}[(x_k)]$ we want to find x_k. To do this we use partial fractions on the right hand side. Recalling the basic Z-tranform $\mathcal{Z}[(r^k)] = z/(z - r)$ we see that we want a z on the top after the partial factions expansion has been done. If we did a partial fraction expansion on $z/(z^2 - z - 2)$ we would end up with something that has no z on top of the fractions. To avoid this we move the z over to the left hand side:

$$\frac{1}{z}X = \frac{1}{z^2 - z - 2} \tag{33}$$

and then write

$$\frac{1}{z^2 - z - 2} = frac1(z-2)(z+1) = \frac{A}{z - 2} + \frac{B}{z + 1} \tag{34}$$

Multiplying across

$$1 = A(z + 1) + B(z - 2) \tag{35}$$

$z = 2$ gives $A = 1/3$ and $z = -1$ gives $B = -1/3$. Now

$$\frac{1}{z}X = \frac{1}{z^2 - z - 2} = \frac{1}{3(z - 2)} - \frac{1}{3(z + 1)} \tag{36}$$

so

$$X = \frac{z}{3(z-2)} - \frac{z}{3(z+1)} \tag{37}$$

and hence

$$x_k = \frac{1}{3}2^k - \frac{1}{3}(-1)^k \tag{38}$$

We can quickly check that this gives the same values as those we calculated above, $k = 0$

$$x_0 = \frac{1}{3} - \frac{1}{3} = 0 \tag{39}$$

$k = 1$

$$x_1 = \frac{1}{3}2 - \frac{1}{3}(-1) = 1 \tag{40}$$

$k = 2$

$$x_2 = \frac{1}{3}4 - \frac{1}{3}(-1)^2 = \frac{4}{3} - \frac{1}{3} = 1 \tag{41}$$

$k = 3$

$$x_2 = \frac{1}{3}8 - \frac{1}{3}(-1)^3 = 3 \tag{42}$$

and $k = 4$

$$x_2 = \frac{1}{3}16 - \frac{1}{3}(-1)^4 = 5 \tag{43}$$

The difference now is that we could also work out $k = 8$ say

$$x_8 = \frac{1}{3}256 - \frac{1}{3} = 85 \tag{44}$$

without having to work out all the lower terms first.

3 Exercises

So far we have studied a difference equation which conveniently has zero on the right hand side and has initial conditions $x_0 = 0$ and $x_1 = 1$. In the next note we will look at examples that aren't quite so convenient, but for now, it is a good idea to pratise some more examples like the one above.

1. Solve the difference equation $x_{k+2} - 4x_{k+1} - 5x_k = 0$ with $x_0 = 0$ and $x_1 = 1$.

2. Solve the difference equation $x_{k+2} - 9x_{k+1} + 20x_k = 0$ with $x_0 = 0$ and $x_1 = 1$.

3. Solve the difference equation $x_{k+2} + 5x_{k+1} + 6x_k = 0$ with $x_0 = 0$ and $x_1 = 1$.

4. Solve the difference equation $x_{k+2} + 2x_{k+1} - 48x_k = 0$ with $x_0 = 0$ and $x_1 = 1$.

5. Solve the difference equation $x_{k+2} + 7x_{k+1} - 18x_k = 0$ with $x_0 = 0$ and $x_1 = 1$.

6. Solve the difference equation $x_{k+2} - 6x_{k+1} + 5x_k = 0$ with $x_0 = 0$ and $x_1 = 1$.

1. So, take the Z-transform of both sides
$$z^2X - z - 4zX - 5X = 0 \tag{45}$$
hence
$$X = \frac{z}{z^2 - 4z - 5} \tag{46}$$
Move the z to the left and do partial fractions,
$$\frac{1}{z}X = \frac{1}{z^2 - 4z - 5} = \frac{1}{(z-5)(z+1)} = \frac{1}{6(z-5)} - \frac{1}{6(z+1)} \tag{47}$$
Thus
$$X = \frac{z}{6(z-5)} - \frac{z}{6(z+1)} \tag{48}$$
and
$$x_k = \frac{1}{6}5^k - \frac{1}{6}(-1)^k \tag{49}$$
2. So, take the Z-transform of both sides
$$z^2X - z - 9zX + 20X = 0 \tag{50}$$
hence
$$X = \frac{z}{z^2 - 9z + 20} \tag{51}$$
Move the z to the left and do partial fractions,
$$\frac{1}{z}X = \frac{1}{z^2 - 9z + 20} = \frac{1}{(z-5)(z-4)} = \frac{1}{z-5} - \frac{1}{z-4} \tag{52}$$
Thus
$$X = \frac{z}{z-5} - \frac{z}{z-4} \tag{53}$$
and
$$x_k = 5^k - 4^k \tag{54}$$
3. So, take the Z-transform of both sides
$$z^2X - z + 5zX + 6X = 0 \tag{55}$$
hence
$$X = \frac{z}{z^2 + 5z + 6} \tag{56}$$
Move the z to the left and do partial fractions,
$$\frac{1}{z}X = \frac{1}{z^2 + 5z + 6} = \frac{1}{(z+2)(z+3)} = \frac{1}{z+2} - \frac{1}{z+3} \tag{57}$$
Thus
$$X = \frac{z}{z+2} - \frac{z}{z+3} \tag{58}$$
and
$$x_k = (-2)^k - (-3)^k \tag{59}$$

4. So, take the Z-transform of both sides
$$z^2X - z + 2zX - 48X = 0 \tag{60}$$
hence
$$X = \frac{z}{z^2 + 2z - 48} \tag{61}$$
Move the z to the left and do partial fractions,
$$\frac{1}{z}X = \frac{1}{z^2 + 2z - 48} = \frac{1}{(z+8)(z-6)} = -\frac{1}{14(z+8)} + \frac{1}{14(z-6)} \tag{62}$$
Thus
$$X = -\frac{z}{14(z+8)} + \frac{z}{14(z-6)} \tag{63}$$
and
$$x_k = -\frac{1}{14}(-8)^k + \frac{1}{14}6^k \tag{64}$$

5. So, take the Z-transform of both sides
$$z^2X - z + 7zX - 18X = 0 \tag{65}$$
hence
$$X = \frac{z}{z^2 + 7z - 18} \tag{66}$$
Move the z to the left and do partial fractions,
$$\frac{1}{z}X = \frac{1}{z^2 + 7z - 18} = \frac{1}{(z-2)(z+9)} = \frac{1}{11(z-2)} - \frac{1}{11(z+9)} \tag{67}$$
Thus
$$X = \frac{z}{11(z-2)} - \frac{z}{11(z+9)} \tag{68}$$
and
$$x_k = \frac{1}{11}(-2)^k - \frac{1}{11}(-9)^k \tag{69}$$

6. So, take the Z-transform of both sides
$$z^2X - z - 6zX + 5X = 0 \tag{70}$$
hence
$$X = \frac{z}{z^2 - 6z + 5} \tag{71}$$
Move the z to the left and do partial fractions,
$$\frac{1}{z}X = \frac{1}{z^2 - 6z + 5} = \frac{1}{(z-5)(z-1)} = \frac{1}{4(z-5)} - \frac{1}{4(z-1)} \tag{72}$$
Thus
$$X = \frac{z}{4(z-5)} - \frac{z}{4(z-1)} \tag{73}$$
and
$$x_k = \frac{1}{4}5^k - \frac{1}{4} \tag{74}$$