Notes on the $Z$-transform, part 2\textsuperscript{1}

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1 The unit pulse

Another entry to the table of standard $Z$-transforms is supplied by the unit pulse

$$\delta_k|_{k=0} = (1, 0, 0, 0, \ldots)$$  \hspace{1cm} (1)

so only the first term of this sequence is non-zero: $x_0 = 1$ and $x_k = 0$ for all $k > 0$. The $Z$-transform of this sequence can be worked out directly, remember the definition of the $Z$-transform

$$Z[\delta_k|_{k=0}] = \sum_{z=0}^{\infty} \frac{x_k}{z^k}$$  \hspace{1cm} (2)

In this case there is only one term in the sequence and hence only one term in the sum:

$$Z[\delta_k|_{k=0}] = \frac{1}{z^0} = 1$$  \hspace{1cm} (3)

2 Linearity of the $Z$-transform

It is possible to add sequences and to multiply them by constants. Say $(x_k)_{k=0}^{\infty}$ and $(y_k)_{k=0}^{\infty}$ are two sequences and $a$ and $b$ are two constants then we have a sequence

$$(ax_k + by_k)_{k=0}^{\infty} = a(x_k)_{k=0}^{\infty} + b(y_k)_{k=0}^{\infty}$$  \hspace{1cm} (4)

For example

$$(2^k + 3)_{k=0}^{\infty} = (4.5, 7, 11, \ldots)$$  \hspace{1cm} (5)

is the sum of the sequence $(2^k) = (1, 2, 4, \ldots)$ and the sequence $(3, 3, 3, \ldots)$. Another example is the sequence $(ar^k)$ which is a times the sequence $(r^k)$. The $Z$-transform is linear. This means that

$$Z[(ax_k + by_k)_{k=0}^{\infty}] = aZ[(x_k)_{k=0}^{\infty}] + bZ[(y_k)_{k=0}^{\infty}]$$  \hspace{1cm} (6)

Hence, for example,

$$Z[(4, 5, 7, 11, \ldots)] = Z[(2^k + 3)_{k=0}^{\infty}] = Z[(2^k)_{k=0}^{\infty}] + 3Z[(1, 1, 1, \ldots)]$$

$$= \frac{4z^2 - 7z}{z^2 - 3z + 2}$$  \hspace{1cm} (7)

Another general example is

$$Z[(ar^k)_{k=0}^{\infty}] = aZ[(r^k)_{k=0}^{\infty}] = \frac{az}{z - r}$$  \hspace{1cm} (8)

3 The delay theorem or first shift theorem

A delayed sequence is one that starts later than some other known sequence. It has the form $(x_{k-k_0})_{k=0}^{\infty}$ so for $k < k_0$ the subscript is negative and, since $x_j = 0$ for $j < 0$, all the entries up to $k = k_0$ are zero. Here is an example, say

$$(x_k)_{k=0}^{\infty} = (4.5, 7, 11, \ldots)$$  \hspace{1cm} (9)

then

$$(x_{k-4})_{k=0}^{\infty} = (0, 0, 4.5, 7, 11, \ldots)$$  \hspace{1cm} (10)

so, the sequence is delayed by $4$ steps. Here is another example,

$$(\delta_{k-4})_{k=0}^{\infty} = (0, 0, 0, 1, 0, 0, 0, \ldots)$$  \hspace{1cm} (11)

In this example the unit pulse has been delayed by four steps.

The delay theorem tells us how to related the $Z$-transform of a delayed sequence to the $Z$-transform of the corresponding undelayed sequence. It states that

$$Z[(x_{k-k_0})] = \frac{1}{z^k}Z[(x_k)]$$  \hspace{1cm} (12)

This theorem is also called the first shift theorem for $Z$-transforms. It is easy enough to prove. By definition

$$Z[(x_{k-k_0})] = \sum_{k=0}^{\infty} \frac{x_{k-k_0}}{z^k}$$  \hspace{1cm} (13)

Now do a change of index: let $k' = k - k_0$. Hence, $k = k' + k_0$; when $k = 0$, $k' = -k_0$ and when $k = \infty$, $k' = \infty$. Thus

$$Z[(x_{k-k_0})] = \sum_{k'=\infty}^{k=\infty} \frac{x_{k'}}{z^{k+k_0}}$$  \hspace{1cm} (14)

However, because $x_{k'} = 0$ for $k' < 0$ the sum starts at $k' = 0$. Thus

$$Z[(x_{k-k_0})] = \sum_{k'=0}^{\infty} \frac{x_{k'}}{z^{k+k_0}} = \frac{1}{z^k} \sum_{k'=0}^{\infty} \frac{x_{k'}}{z^{k'}} = \frac{1}{z^k}Z[(x_k)]$$  \hspace{1cm} (15)

as required.

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Here is an example: the sequence \((0, 0, 1, 3, 9, 27, \ldots) = (x_{k-2})_{k=0}^\infty\) where \((x_k)_{k=0}^\infty = (3^k)_{k=0}^\infty = (1, 3, 9, 27, \ldots)\) so the Z-transform is
\[
\mathcal{Z}[0, 0, 1, 3, 9, 27, \ldots] = \frac{1}{z^2}\mathcal{Z}[1, 3, 9, 27, \ldots] = \frac{1}{z^2} \frac{z}{z^2 - 3}
\]
(16)

Here is another example, consider
\[
\mathcal{Z}[(\delta_{k-1})_{k=0}^\infty] = \frac{1}{z^2}\mathcal{Z}[\delta_k] = \frac{1}{z^2}
\]
(17)
since \(\mathcal{Z}[\delta_k] = 1\). The sequence \((0, 1, 1, 1, 1, \ldots)\) is the one step delay of \((1, 1, 1, 1, \ldots)\) so we have
\[
\mathcal{Z}[0, 1, 1, 1, 1, \ldots] = \frac{1}{z}\mathcal{Z}[1, 1, 1, 1, \ldots] = \frac{1}{z} \frac{z}{z - 1} = \frac{1}{z - 1}
\]
(18)

There is another way of doing the same question: by using linearity. We can write
\[
0, 1, 1, 1, 1, \ldots = (1, 1, 1, 1, \ldots) - (0, 0, 0, 0, \ldots)
\]
(19)

so
\[
\mathcal{Z}[0, 1, 1, 1, 1, \ldots] = \mathcal{Z}[1, 1, 1, 1, \ldots] - \mathcal{Z}[1, 0, 0, 0, \ldots]
\]
\[
= \frac{z}{z - 1} - 1 = \frac{z - (z - 1)}{z - 1} = \frac{1}{z - 1}
\]
(20)

4 Exercises

Work out the Z-transform of the following sequences

1. \((2, 4, 10, 28, \ldots)\)
2. \((-2, 10, -26, \ldots)\)
3. \((3, 0, 0, 0, \ldots)\)
4. \((0, 0, 1, 1, 1, \ldots)\)
5. \((0, 2, 4, 10, 28, \ldots)\)
6. \((0, 0, 1, 2, 4, 8, \ldots)\)
7. \((1, 1, 0, 1, 1, 1, \ldots)\)

1. So this sequence is
\[
(2, 4, 10, 28, \ldots) = (1, 1, 1, \ldots) + (1, 3, 9, 27, \ldots)
\]
so we use linearity
\[
\mathcal{Z}[2, 4, 10, 28, \ldots] = \mathcal{Z}[1, 1, 1, \ldots] + \mathcal{Z}[1, 3, 9, 27, \ldots]
\]
\[
= \frac{z}{z - 1} + \frac{z}{z - 1} = \frac{2z}{z - 1} - \frac{4z^2 - 4z}{z^2 - 4z + 3}
\]
(22)

2. So this sequence is
\[
(-2, -10, -26, \ldots) = (1, 1, 1, \ldots) + (1, -3, 9, -27, \ldots)
\]
so we use linearity
\[
\mathcal{Z}[(-2, -10, -26, \ldots)] = \mathcal{Z}[1, 1, 1, \ldots] + \mathcal{Z}[1, -3, 9, -27, \ldots]
\]
\[
= \frac{z}{z - 1} + \frac{z}{z - 1} = \frac{2z^2 + 2z}{z^2 - 4z + 3}
\]
(24)

3. \((3, 0, 0, 0, \ldots) = 3(\delta_k)\) so \(\mathcal{Z}[(3, 0, 0, 0, \ldots)] = 3\).

4. \((0, 0, 1, 1, 1, \ldots)\) is the \(k_0 = 2\) delay of \((1, 1, 1, 1, \ldots)\) which means that
\[
\mathcal{Z}[(0, 0, 1, 1, 1, \ldots)] = \frac{1}{z^2} \frac{1}{z - 1}
\]
(25)

5. \((0, 2, 4, 10, 28, \ldots) = (x_{k-1})\) where \((x_k) = (2, 4, 10, 28, \ldots)\) as in first exercise, hence,
\[
\mathcal{Z}[(0, 2, 4, 10, 28, \ldots)] = \frac{1}{z^2} \frac{2z^2 - 4z}{z^2 - 4z + 3} = \frac{2z}{z^2 - 4z + 3}
\]
(26)

6. \((0, 0, 1, 2, 4, 8, \ldots) = (2^{k-2})\) and \(\mathcal{Z}[(2^k)] = \frac{z}{z - 2}\), so,
\[
\mathcal{Z}[(0, 0, 1, 2, 4, 8, \ldots)] = \frac{1}{z^2} \frac{z}{z - 2} = \frac{1}{z^2 - z - 2}
\]
(27)

7. This one is a bit trickier. Notice that
\[
(1, 1, 0, 1, 1, \ldots) = (1, 1, 1, 1, \ldots) - (0, 0, 1, 0, 0, \ldots)
\]
and \((0, 0, 1, 0, 0, \ldots) = (\delta_{k-2})\). Hence, using linearity and the delay theorem we get
\[
\mathcal{Z}[(1, 1, 0, 1, 1, \ldots)] = \mathcal{Z}[(1, 1, 1, 1, \ldots)] - \mathcal{Z}[(0, 0, 1, 0, 0, \ldots)]
\]
\[
= \frac{z}{z - 1} - \frac{1}{z^2} = \frac{z^3 - z + 1}{z^2(z - 1)}
\]
(29)