Notes on the Z-transform, part 2[1]

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1 The unit pulse

Another entry to the table of standard Z-tranforms is supplied by the unit pulse

$$(\delta_k)_{k=0}^{\infty} = (1, 0, 0, 0, \ldots) \tag{1}$$

so only the first term of this sequence is non-zero: $x_0 = 1$ and $x_k = 0$ for all $k > 0$. The Z-transform of this sequence can be worked out directly, remember the definition of the Z-tranform

$$\mathcal{Z}[(x_k)_{k=0}^{\infty}] = \sum_{z=0}^{\infty} \frac{x_k}{z^k}. \tag{2}$$

In this case there is only one term in the sequence and hence only one term in the sum:

$$\mathcal{Z}[(\delta_k)_{k=0}^{\infty}] = \frac{1}{z^0} = 1 \tag{3}$$

2 Linearity of the Z-tranform

It is possible to add sequences and to multiply them by constants. Say $(x_k)_{k=0}^{\infty}$ and $(y_k)_{k=0}^{\infty}$ are two sequences and a and b are two constants then we have a sequence

$$(ax_k + by_k)_{k=0}^{\infty} = a(x_k)_{k=0}^{\infty} + b(y_k)_{k=0}^{\infty} \tag{4}$$

For example

$$(2^k + 3)_{k=0}^{\infty} = (4, 5, 7, 11, \ldots) \tag{5}$$

is the sum of the sequence $(2^k) = (1, 2, 4, 8, \ldots)$ and the sequence $(3, 3, 3, \ldots)$. Another example is the sequence (ar^k) which is a times the sequence (r^k).

The Z-tranform is linear. This means that

$$\mathcal{Z}[(ax_k + by_k)_{k=0}^{\infty}] = a\mathcal{Z}[(x_k)_{k=0}^{\infty})] + b\mathcal{Z}[(y_k)_{k=0}^{\infty}] \tag{6}$$

Hence, for example,

$$\begin{aligned}
\mathcal{Z}[(4, 5, 7, 11, \ldots)] &= \mathcal{Z}[(2^k + 3)_{k=0}^{\infty}] = \mathcal{Z}[(2^k)_{k=0}^{\infty}] + 3\mathcal{Z}[(1, 1, 1, \ldots)] \\
&= \frac{z}{z-2} + \frac{3z}{z-1} = \frac{z(z-1) + 3z(z-2)}{(z-2)(z-1)}
\end{aligned}$$

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$$= \frac{4z^2 - 7z}{z^2 - 3z + 2} \tag{7}$$

Another general example is

$$\mathcal{Z}[(ar^k)_{k=0}^{\infty}] = a\mathcal{Z}[(r^k)_{k=0}^{\infty}] = \frac{az}{z-r} \tag{8}$$

3 The delay theorem or first shift thereom

A delayed sequence is one that starts later than some other known sequence. It has the form $(x_{k-k_0})_{k=0}^{\infty}$ so for $k < k_0$ the subscipt is negative and, since $x_j = 0$ for $j < 0$, all the entries up to $k = k_0$ are zero. Here is an example, say

$$(x_k)_{k=0}^{\infty} = (4, 5, 7, 11, \ldots) \tag{9}$$

then

$$(x_{k-2})_{k=0}^{\infty} = (0, 0, 4, 5, 7, 11, \ldots) \tag{10}$$

so, the sequence is delayed by $k_0 = 2$ steps. Here is another example,

$$(\delta_{k-4})_{k=0}^{\infty} = (0, 0, 0, 0, 1, 0, 0, 0, \ldots) \tag{11}$$

In this example the unit pulse has been delayed by four steps.

The delay theorem tells us how to related the Z-tranform of a delayed sequence to the Z-tranform of the corresponding undelayed sequence. It states that

$$\mathcal{Z}[(x_{k-k_0})] = \frac{1}{z^{k_0}}\mathcal{Z}[(x_k)] \tag{12}$$

This theorem is also called the first shift theorem for Z-transforms. It is easy enough to prove. By definition

$$\mathcal{Z}[(x_{k-k_0})] = \sum_{k=0}^{\infty} \frac{x_{k-k_0}}{z^k} \tag{13}$$

Now do a change of inedex: let $k' = k - k_0$. Hence, $k = k' + k_0$; when $k = 0$, $k' = -k_0$ and when $k = \infty$, $k' = \infty$. Thus

$$\mathcal{Z}[(x_{k-k_0})] = \sum_{k'=-k_0}^{\infty} \frac{x_{k'}}{z^{k'+k_0}} \tag{14}$$

However, because $x_{k'} = 0$ for $k' < 0$ the sum starts at $k' = 0$. Thus

$$\mathcal{Z}[(x_{k-k_0})] = \sum_{k'=0}^{\infty} \frac{x_{k'}}{z^{k'+k_0}} = \frac{1}{z^{k_0}} \sum_{k'=0}^{\infty} \frac{x_{k'}}{z^{k'}} = \frac{1}{z^{k_0}}\mathcal{Z}[(x_k)] \tag{15}$$

as required.

Here is an example: the sequence $(0, 0, 1, 3, 9, 27, \ldots) = (x_{k-2})_{k=0}^{\infty}$ where $(x_k)_{k=0}^{\infty} = (3^k)_{k=0}^{\infty} = (1, 3, 9, 27, \ldots)$ so the Z-tranform is

$$\mathcal{Z}[(0, 0, 1, 3, 9, 27, \ldots)] = \frac{1}{z^2}\mathcal{Z}[(1, 3, 9, 27, \ldots)] = \frac{1}{z^2}\frac{z}{z-3} \tag{16}$$

Here is another example, consider

$$\mathcal{Z}[(\delta_{k-4})_{k=0}^{\infty}] = \frac{1}{z^4}\mathcal{Z}[(\delta_k)] = \frac{1}{z^4} \tag{17}$$

since $\mathcal{Z}[(\delta_k)] = 1$. The seqence $(0, 1, 1, 1, 1, \ldots)$ is the one step delay of $(1, 1, 1, 1, \ldots)$ so we have

$$\mathcal{Z}[(0, 1, 1, 1, 1, \ldots)] = \frac{1}{z}\mathcal{Z}[(1, 1, 1, 1, \ldots)] = \frac{1}{z}\frac{z}{z-1} = \frac{1}{z-1} \tag{18}$$

There is anther way of doing the same question: by using linearity. We can write

$$(0, 1, 1, 1, 1, \ldots) = (1, 1, 1, 1, \ldots) - (1, 0, 0, 0, \ldots) \tag{19}$$

so

$$\begin{aligned} \mathcal{Z}[(0, 1, 1, 1, 1, \ldots)] &= \mathcal{Z}[(1, 1, 1, 1, \ldots)] - \mathcal{Z}[(1, 0, 0, 0, \ldots)] \\ &= \frac{z}{z-1} - 1 = \frac{z-(z-1)}{z-1} = \frac{1}{z-1} \end{aligned} \tag{20}$$

4 Exercises

Work out the Z-transform of the following sequences

1. $(2, 4, 10, 28, \ldots)$
2. $(2, -2, 10, -26, \ldots)$
3. $(3, 0, 0, 0, \ldots)$
4. $(0, 0, 1, 1, 1, \ldots)$
5. $(0, 2, 4, 10, 28, \ldots)$
6. $(0, 0, 1, 2, 4, 8, \ldots)$
7. $(1, 1, 0, 1, 1, 1, \ldots)$

1. So this sequence is

$$(2, 4, 10, 28, \ldots) = (1, 1, 1, 1, \ldots) + (1, 3, 9, 27, \ldots) \tag{21}$$

so we use linearity

$$\begin{aligned}\mathcal{Z}[(2, 4, 10, 28, \ldots)] &= \mathcal{Z}[(1, 1, 1, 1, \ldots)] + \mathcal{Z}[(1, 3, 9, 27, \ldots)] \\ &= \frac{z}{z-1} + \frac{z}{z-3} = \frac{2z^2 - 4z}{z^2 - 4z + 3}\end{aligned} \tag{22}$$

2. So this sequence is

$$(2, -2, 10, -26, \ldots) = (1, 1, 1, 1, \ldots) + (1, -3, 9, -27, \ldots) \tag{23}$$

so we use linearity

$$\begin{aligned}\mathcal{Z}[(2, -2, 10, -27, \ldots)] &= \mathcal{Z}[(1, 1, 1, 1, \ldots)] + \mathcal{Z}[(1, -3, 9, -27, \ldots)] \\ &= \frac{z}{z-1} + \frac{z}{z+3} = \frac{2z^2 + 2z}{z^2 + 2z - 3}\end{aligned} \tag{24}$$

3. $(3, 0, 0, 0, \ldots) = 3(\delta_k)$ so $\mathcal{Z}[(3, 0, 0, 0, \ldots)] = 3$.

4. $(0, 0, 1, 1, 1, \ldots)$ is the $k_0 = 2$ delay of $(1, 1, 1, 1, \ldots)$ which means that

$$\mathcal{Z}[(0, 0, 1, 1, 1, \ldots)] = \frac{1}{z^2} \frac{z}{z-1} \tag{25}$$

5. $(0, 2, 4, 10, 28, \ldots)$ is (x_{k-1}) where $(x_k) = (2, 4, 10, 28, \ldots)$ as in first exercise, hence,

$$\mathcal{Z}[(0, 2, 4, 10, 28, \ldots)] = \frac{1}{z} \frac{2z^2 - 4z}{z^2 - 4z + 3} = \frac{2z - 4}{z^2 - 4z + 3} \tag{26}$$

6. $(0, 0, 1, 2, 4, 8, \ldots) = (2^{k-2})$ and $\mathcal{Z}[(2^k)] = z/(z-2)$, so,

$$\mathcal{Z}[(0, 0, 1, 2, 4, 8, \ldots)] = \frac{1}{z^2} \frac{z}{z-2} = \frac{1}{z(z-2)} \tag{27}$$

7. This one is a bit trickier. Notice that

$$(1, 1, 0, 1, 1, 1, \ldots) = (1, 1, 1, 1, 1, \ldots) - (0, 0, 1, 0, 0, \ldots) \tag{28}$$

and $(0, 0, 1, 0, 0, \ldots) = (\delta_{k-2})$. Hence, using linearity and the delay theorem we get

$$\begin{aligned}\mathcal{Z}[(1, 1, 0, 1, 1, 1, \ldots)] &= \mathcal{Z}[(1, 1, 1, 1, 1, \ldots)] - \mathcal{Z}[(0, 0, 1, 0, 0, \ldots)] \\ &= \frac{z}{z-1} - \frac{1}{z^2} = \frac{z^3 - z + 1}{z^2(z-1)}\end{aligned} \tag{29}$$