Notes on the Z-transform, part 1[1]

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1 Introduction

The Z-transform is a tranform for sequences. Just like the Laplace transform takes a function of t and replaces it with another function of an auxillary variable s, well, the Z-transform takes a sequence and replaces it with a function of an auxillary variable, z. The reason for doing this is that it makes difference equations easier to solve, again, this is very like what happens with the Laplace tranform, where taking the Laplace tranform makes it easier to solve differential equations. A difference equation is an equation which tells you what the $k+2$th term in a sequence is in terms of the $k+1$th and kth terms, for example. Difference equations arise in numerical treatments of differential equations, in discrete time sampling and when studying systems that are intrinsically discrete, such as population models in ecology and epidemiology and mathematical modelling of mylinated nerves.

2 Review of Sequences

A sequence is a list of numbers, sequences can be finite, like $(2, 2, 3, 4)$ or infinite, like $(1, 2, 3, 4, 5, \ldots)$. We are interested in infinite sequences. These all have the general form $(x_0, x_1, x_3, \ldots)$ with the x_ks standing for the numbers in the sequence. We use the short hand

$$(x_k)_{k=0}^{\infty} = (x_0, x_1, x_2, \ldots) \tag{1}$$

In otherwords, on the righthand side, we are saying the sequence is formed by writing out the x_ks with k put equal to zero, then one and so on up to infinity. Often we are lazy and just write (x_k) when we mean$(x_k)_{k=0}^{\infty}$.

The sequence

$$(2, 5, 8, 11, 14, \ldots) \tag{2}$$

is an arithmetic sequence, each term is calculated by adding three to the term before it. In fact, you can write a formula telling you what the kth term is

$$x_k = 2 + 3k \tag{3}$$

and so

$$(2 + 3k)_{k=0}^{\infty} = (2, 5, 8, 11, 14, \ldots) \tag{4}$$

[1]Conor Houghton, houghton@maths.tcd.ie, see also http://www.maths.tcd.ie/~houghton/2E2.html

This is really the most you can know about the sequence, if you have a formula saying x_k is such and such a thing involving k then you have solved the sequence and you can easily work out x_k for any k, for example, we know that $x_{1000} = 3002$ in the example above. Another way of writing down the information in the sequence is to say

$$\begin{aligned} x_{k+1} &= x_k + 3 \\ x_0 &= 2 \end{aligned} \tag{5}$$

This tells you how to find the next term in the sequence in terms of the previous one and it tells you where to start, at $x_0 = 2$. This is a difference equation.[2] It allows you to work out the sequence, but only step by step. To calculate x_{1000} you would first of all have to know x_{999} and to work this out you need x_{998} and so on and so on. We will see later how to use Z-tranforms to solve a sequence when we only know the difference equation.

Another common sort of sequence is a geometric sequence, an example is

$$(3, 15, 75, 375, \ldots) \tag{6}$$

Here, each term is given by multiplying the previous term by five, so, the difference equation is

$$\begin{aligned} x_{k+1} &= 5x_k \\ x_0 &= 3 \end{aligned} \tag{7}$$

We also know the solution in this case,

$$x_k = 3 \times 5^k \tag{8}$$

That is

$$(3 \times 5^k)_{k=0}^{\infty} = (3, 15, 75, 375, \ldots) \tag{9}$$

More generally, a geometric sequence has the form

$$(ar^k)_{k=0}^{\infty} = (a, ar, ar^2, ar^3, \ldots) \tag{10}$$

and $x_0 = a$ and r is called the ratio. Thus,

$$\left(60, 30, 15, \frac{15}{2}, \frac{15}{4}, \ldots\right) \tag{11}$$

is a geometrical sequence with $a = 60$ and $r = 1/2$. The sequence

$$(1, 1, 1, 1, 1, \ldots) \tag{12}$$

is a geometrical sequence with $a = 1$ and $r = 1$.

[2] There is another name also used for a difference equation, it can be called an induction step, the difference usually is that you called it a difference equation if you intend to solve it and an induction step if you intend to use it as it is to work out the sequence term by term.

A series is what you get when you sum up all the terms in a sequence. Consider the sequence

$$\left(\frac{1}{2^k}\right)_{k=0}^{\infty} = \left(1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\right) \tag{13}$$

The corresponding sum is

$$S = \sum_{k=0}^{\infty} \frac{1}{2^k} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots \tag{14}$$

and if you think about it, $S = 2$ since, at each step you are adding half the distance from where you are to two, if you keep adding for infinity you will finally get to two.[3] In fact, you can sum any geometric sequence with ratio less than one. Let

$$S = \sum_{k=0}^{\infty} ar^k = a \sum_{k=0}^{\infty} r^k \tag{15}$$

Next, we remember the formula that

$$\frac{1}{1-r} = \sum_{k=0}^{\infty} r^k \tag{16}$$

for $|r| < 1$. You can check this formula either using the Taylor series or the binomial expansion. Hence

$$S = \frac{a}{1-r} \tag{17}$$

If the ratio is bigger than or equal to one then the corresponding series diverges, that is, it doesn't sum up to finite amount.

3 The Z-transform

The Z-transform of a sequence (x_k) is defined as

$$\mathcal{Z}[(x_k)_{k=0}^{\infty}] = \sum_{k=0}^{infty} \frac{x_k}{z^k} \tag{18}$$

We often write

$$X(z) = \mathcal{Z}[(x_k)_{k=0}^{\infty}] \tag{19}$$

[3]Zeno's paradox was based on this, Zeno, a philosopher in ancient Greece, worried about the fact that to get from A to B you had to first cross half the distance from A to B, then half of the distance remaining and so on for infinity. He thought that, since you had to cross an infinity of shorter and shorter distances, it must take you an infinite amount of time. He is wrong, of course, each shorter and shorter distance takes less and less time to cross and it all adds up to a finite number. Zeno had two paradoxes, the other has to do with asking when something moves, since at any instant it is in just one place.

You can see that when you do the Z-transform it sums up all the sequence, and so the individual terms affect the dependence on z, but the resulting function is just a function of z, it has no k in it. It will become clearer later why we might do this, first, we will look at how to calculate the Z-tranform of a few example.

Lets find the Z-tranform of a geometric sequence (r^k). We have

$$\mathcal{Z}[(r^k)_{k=0}^{\infty}] = \sum_{k=0}^{infty} \frac{r^k}{z^k} = \sum_{k=0}^{infty} \left(\frac{r}{z}\right)^k = \frac{1}{1-\frac{r}{z}} = \frac{z}{z-r} \tag{20}$$

In particular, this means that

$$\mathcal{Z}[(1,1,1,\ldots)] = \frac{z}{z-1} \tag{21}$$

Another Z-transform can be derived from this by differenciating with repect to r. This is a usefull trick, you might worry about whether it is okay to differenciate through the sum and so on, but we just assume everything works. So,

$$\begin{aligned}
\mathcal{Z}[(r^k)_{k=0}^{\infty}] &= \frac{z}{z-r} \\
\frac{d}{dr}\mathcal{Z}[(r^k)_{k=0}^{\infty}] &= \frac{d}{dr}\frac{z}{z-r} \\
\mathcal{Z}\left[\frac{d}{dr}(r^k)_{k=0}^{\infty}\right] &= \frac{z}{(z-r)^2} \\
\mathcal{Z}[(kr^{k-1})_{k=0}^{\infty}] &= \frac{z}{(z-r)^2}
\end{aligned} \tag{22}$$

and, in particular, this means

$$\mathcal{Z}[(0,1,2,3,\ldots)] = \frac{z}{(z-1)^2} \tag{23}$$

Hence, we now have two entries in our table of Z-transforms

$$\begin{aligned}
\mathcal{Z}[(r^k)_{k=0}^{\infty}] &= \frac{z}{z-r} \\
\mathcal{Z}[(kr^{k-1})_{k=0}^{\infty}] &= \frac{z}{(z-r)^2}
\end{aligned} \tag{24}$$

4 Exercises

Find the Z-tranforms of

$$\begin{array}{lll} (a)\ \left(\frac{1}{4^k}\right) & (b)\ (3^k) & (c)\ ((-2)^k) \\ (d)\ (4,16,64,256,\ldots) & (e)\ (1,-3,9,-27,\ldots) & (f)\ (0,1,4,12,64,160,\ldots) \end{array} \tag{25}$$

For (a) we have $r = 1/4$ so

$$\mathcal{Z}\left[\left(\frac{1}{4^k}\right)\right] = \frac{z}{z-1/4} = \frac{4z}{4z-1} \tag{26}$$

For (b) $r = 3$ giving For (a) we have $r = 1/4$ so

$$\mathcal{Z}[(3^k) = \frac{z}{z-3} \tag{27}$$

In (c) $r = -2$ but this makes no difference

$$\mathcal{Z}[((-2)^k)] = \frac{z}{z+2} \tag{28}$$

In (d) we see that $r = 4$ so

$$\mathcal{Z}[(4,16,64,256,\ldots)] = \frac{z}{z-4} \tag{29}$$

and in (e) $r = -3$ so

$$\mathcal{Z}[(1,-3,9,-27,\ldots)] = \frac{z}{z+3} \tag{30}$$

Finally, looking carefully at (f) you realize

$$(k2^{k-1}) = (0,1,4,12,64,160,\ldots) \tag{31}$$

and, hence,

$$\mathcal{Z}[(0,1,4,12,64,160,\ldots)] = \frac{z}{(z-2)^2} \tag{32}$$