2E2 Tutorial Sheet 9 Second Term, Solutions[1]

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1. (3) Find the eigenvectors and eigenvalues of the following matrices

$$(i)\quad \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix} \qquad (ii)\quad \begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix} \qquad (iii)\quad \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \tag{1}$$

Solution:

In (i) the characteristic equation is

$$\begin{vmatrix} 3-\lambda & 4 \\ 4 & -3-\lambda \end{vmatrix} = 0 \tag{2}$$

so

$$(3-\lambda)(-3-\lambda) - 16 = 0 \tag{3}$$

or

$$\lambda^2 + 2\lambda - 48 - 25 = 0 \tag{4}$$

Solve this gives us $\lambda = \pm 5$. Taking the $\lambda = 5$ first

$$\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 5 \begin{pmatrix} a \\ b \end{pmatrix} \tag{5}$$

so the first equation is $3a + 4b = 5a$ or $a = 2b$, the other equation is $4a - 3b = 5b$ which is also $a = 2b$. Taking $a = 2$ an eigenvalue 5 eigenvector is,

$$\mathbf{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{6}$$

Taking $\lambda = -5$ next

$$\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = -5 \begin{pmatrix} a \\ b \end{pmatrix} \tag{7}$$

so the first equation is $3a+4b = -5a$ or $2a = -b$, the other equation is $4a-3b = -5b$ which is also $2a = -b$. Taking $a = 1$ an eigenvalue -5 eigenvector is,

$$\mathbf{x} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \tag{8}$$

In (ii) the characteristic equation is

$$\begin{vmatrix} -\lambda & 3 \\ -3 & -\lambda \end{vmatrix} = 0 \tag{9}$$

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so

$$\lambda^2 + 9 = 0 \tag{10}$$

or

$$\lambda = \pm 3i \tag{11}$$

Taking the $\lambda = 3i$ first

$$\begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 3i \begin{pmatrix} a \\ b \end{pmatrix} \tag{12}$$

so the equation is $3b = 3ia$ or $a = -ib$. Taking $b = 1$ an eigenvalue $3i$ eigenvector is,

$$\mathbf{x} = \begin{pmatrix} -i \\ 1 \end{pmatrix} \tag{13}$$

Taking the $\lambda = -3i$

$$\begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = -3i \begin{pmatrix} a \\ b \end{pmatrix} \tag{14}$$

so the equation is $3b = -3ia$ or $a = ib$. Taking $a = 1$ an eigenvalue $-3i$ eigenvector is,

$$\mathbf{x} = \begin{pmatrix} i \\ 1 \end{pmatrix} \tag{15}$$

In (iii) the characteristic equation is

$$\begin{vmatrix} 1-\lambda & 2 \\ 0 & 3-\lambda \end{vmatrix} = 0 \tag{16}$$

so

$$(1-\lambda)(3-\lambda) = 0 \tag{17}$$

So this gives us $\lambda = 1$ or $\lambda = 3$. Taking the $\lambda = 1$ first

$$\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \tag{18}$$

so the first equation is $a + 2b = a$ or $0 = b$, the other equation is $3b = b$ which is also $b = 0$. Taking $a = 1$ an eigenvalue 1 eigenvector is,

$$\mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{19}$$

Taking $\lambda = 3$ next

$$\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 3 \begin{pmatrix} a \\ b \end{pmatrix} \tag{20}$$

so the first equation is $a + 2b = 3a$ or $a = b$, the other equation is $3b = 3b$ which tells us nothing. Taking $a = 1$ an eigenvalue 3 eigenvector is,

$$\mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{21}$$

2. (3) Find the solution for the system

$$\begin{aligned}\frac{dy_1}{dt} &= -3y_1 + 2y_2 \\ \frac{dy_2}{dt} &= -2y_1 + 2y_2\end{aligned}$$

This equation is $\mathbf{y}' = A\mathbf{y}$ with

$$A = \begin{pmatrix} -3 & 2 \\ -2 & 2 \end{pmatrix}$$

We can find the eigenvalues, the characteristic equation is

$$\begin{vmatrix} -3-\lambda & 2 \\ -2 & 2-\lambda \end{vmatrix} = (\lambda+3)(\lambda-2)+4 = \lambda^2+\lambda-2 = 0$$

so that $\lambda_1 = 1$ and $\lambda_2 = -2$.

Next, we need the eigenvectors. First, λ_1:

$$\begin{pmatrix} -3 & 2 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \tag{22}$$

so $-3a + 2b = a$ or $b = 2a$, hence, choosing $a = 1$ we get

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}. \tag{23}$$

For λ_2:

$$\begin{pmatrix} -3 & 2 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = -2\begin{pmatrix} a \\ b \end{pmatrix} \tag{24}$$

so $-3a + 2b = -2a$ giving $a = 2b$, choosing $b = 1$ gives

$$\mathbf{x}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{25}$$

Now, in general the solution is

$$\mathbf{y} = c_1\mathbf{x}_1 e^{\lambda_1 t} + c_2\mathbf{x}_2 e^{\lambda_2 t} \tag{26}$$

so, here,

$$\mathbf{y} = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{t} + c_2\begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-2t} \tag{27}$$

3. (2) Find the general solutions for the system

$$\frac{dy_1}{dt} = 3y_1 + y_2 \tag{28}$$

$$\frac{dy_2}{dt} = y_1 + 3y_2 \tag{29}$$

Solution: The eigenvectors and eigenvalues of

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \tag{30}$$

are $\lambda_1 = 4$ with

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{31}$$

and $\lambda_2 = 2$ with

$$\mathbf{x}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{32}$$

so the general soln is

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{2t}. \tag{33}$$