2E2 Tutorial Sheet 8 Second Term[1]

28 November 2003

1. (3) Find the eigenvectors and eigenvalues of the following matrices

$$(i)\quad \begin{pmatrix} 4 & 0 \\ 0 & -6 \end{pmatrix} \qquad (ii)\quad \begin{pmatrix} 10 & -4 \\ 18 & -12 \end{pmatrix} \qquad (iii)\quad \begin{pmatrix} 0 & r \\ r & 0 \end{pmatrix} \tag{1}$$

Solution: In (i) the characteristic equation is

$$\begin{vmatrix} 4-\lambda & 0 \\ 0 & -6-\lambda \end{vmatrix} = 0 \tag{2}$$

so

$$(4-\lambda)(-6-\lambda) = 0 \tag{3}$$

So $\lambda = 4$ or $lambda = -6$. Taking the $\lambda = 4$ first

$$\begin{pmatrix} 4 & 0 \\ 0 & -6 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 4\begin{pmatrix} a \\ b \end{pmatrix} \tag{4}$$

so $4a = 4a$ and $-6b = 4b$, hence $b = 0$ and a is arbitrary, taking $a = 1$ an eigenvalue 4 eigenvector is,

$$\mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{5}$$

Taking the $\lambda = -6$

$$\begin{pmatrix} 4 & 0 \\ 0 & -6 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = -6\begin{pmatrix} a \\ b \end{pmatrix} \tag{6}$$

so $4a = -6a$ and $-6b = -6b$, hence $a = 0$ and b is arbitrary, taking $b = 1$ a eigenvalue -6 eigenvector is,

$$\mathbf{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{7}$$

In (ii) the characteristic equation is

$$\begin{vmatrix} 10-\lambda & -4 \\ 18 & -12-\lambda \end{vmatrix} = 0 \tag{8}$$

so

$$(10-\lambda)(-12-\lambda) - (-4)18 = 0 \tag{9}$$

[1] Conor Houghton, `houghton@maths.tcd.ie`, see also `http://www.maths.tcd.ie/~houghton/2E2.html`

or

$$\lambda^2 + 2\lambda - 48 = 0 \tag{10}$$

Solve this gives us $\lambda = 6$ or $\lambda = -8$. Taking the $\lambda = 6$ first

$$\begin{pmatrix} 10 & -4 \\ 18 & -12 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 6 \begin{pmatrix} a \\ b \end{pmatrix} \tag{11}$$

so the first equation is $10a - 4b = 6a$ or $a = b$, the other equation is $18a - 12b = 6b$ which is also $a = b$. Taking $a = 1$ an eigenvalue 6 eigenvector is,

$$\mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{12}$$

Taking the $\lambda = -8$ next

$$\begin{pmatrix} 10 & -4 \\ 18 & -12 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = -8 \begin{pmatrix} a \\ b \end{pmatrix} \tag{13}$$

so the first equation is $10a - 4b = -8a$ or $9a = 2b$, the other equation is $18a - 12b = -8b$ which is also $9a = 2b$. Taking $a = 2$, $b = 9$ an eigenvalue -8 eigenvector is,

$$\mathbf{x} = \begin{pmatrix} 2 \\ 9 \end{pmatrix} \tag{14}$$

In (iii) the characteristic equation is

$$\begin{vmatrix} -\lambda & r \\ r & -\lambda \end{vmatrix} = 0 \tag{15}$$

so

$$\lambda^2 - r^2 = 0 \tag{16}$$

or

$$\lambda = \pm r \tag{17}$$

Taking the $\lambda = r$ first

$$\begin{pmatrix} 0 & r \\ r & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = r \begin{pmatrix} a \\ b \end{pmatrix} \tag{18}$$

so the equation is $rb = ra$ or $a = b$. Taking $a = 1$ an eigenvalue r eigenvector is,

$$\mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{19}$$

Taking the $\lambda = -r$

$$\begin{pmatrix} 0 & r \\ r & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = -r \begin{pmatrix} a \\ b \end{pmatrix} \tag{20}$$

so the equation is $rb = -ra$ or $a = -b$. Taking $a = 1$ an eigenvalue $-r$ eigenvector is,

$$\mathbf{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{21}$$

2. (2) Rewrite

$$\begin{aligned} 10y_1 - 4y_2 &= 2 \\ 18y_1 - 12y_2 &= 3 \end{aligned} \tag{22}$$

in the matrix form $A\mathbf{y} = \mathbf{a}$ where

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \tag{23}$$

Solution: This is equivalent to the matrix equation

$$\begin{pmatrix} 10 & -4 \\ 18 & -12 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \tag{24}$$

You can check this by multiplying it out.

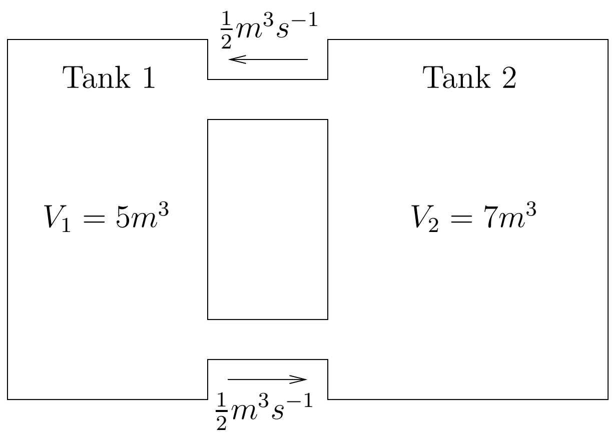

Figure 1: Two containers with flow between them.

3. (3) As illustrated in Fig. 1, two large containers are connected and American style sandwich spead is pumped between them at a rate of $1/2m^3s^{-1}$. One container has volume $5m^3$, the other $7m^3$. Both are full of spread. Initially the smaller container contains pure jam, the second container has $5m^3$ of jam and $2m^3$ of peanut butter. Assume perfect mixing and so on.
(i) Write down the differential equation for $y_1(t)$ and $y_2(t)$, the amount of peanut butter in the first and second container.
(ii) Solve it to find $y_1(t)$ and $y_2(t)$ explicitly.
(iii) The amount of peanut butter in each container will change less and less as time passes, what values are the tending towards.

Solution: Well if there is y_1 peanut butter in the small container then the concentration of the spead in the small container is $y_1/5$ and so $y_1/10$ is flowing out per second. In the same way $y_2/7$ is the concentration of peanut butter in the second tank and so $y_2/14$ per second is going from the large tank to the small one. This means the equations are

$$y_1' = -\frac{1}{10}y_1 + \frac{1}{14}y_2 \quad (25)$$

$$y_2' = \frac{1}{10}y_1 - \frac{1}{14}y_2 \quad (26)$$

This equation can be rewritten

$$\mathbf{y}' = \begin{pmatrix} -\frac{1}{10} & \frac{1}{14} \\ \frac{1}{10} & -\frac{1}{14} \end{pmatrix} \mathbf{y} \quad (27)$$

We work out the eigenvalues

$$\begin{vmatrix} -\frac{1}{10} - \lambda & \frac{1}{14} \\ \frac{1}{10} & -\frac{1}{14} - \lambda \end{vmatrix} = \left(\frac{1}{10} + \lambda\right)\left(\frac{1}{14} + \lambda\right) - \frac{1}{140} \quad (28)$$

$$= \lambda^2 + \frac{6}{35}\lambda = 0 \quad (29)$$

This means that there are two eigenvalues, $\lambda_1 = 0$ and $\lambda_2 = -6/35$. The corresponding eigenvectors are given by

$$\begin{pmatrix} -\frac{1}{10} & \frac{1}{14} \\ \frac{1}{10} & -\frac{1}{14} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0 \quad (30)$$

which has solutions of the form $a = 10$ and $b = 14$ for λ_1 and

$$\begin{pmatrix} -\frac{1}{10} & \frac{1}{14} \\ \frac{1}{10} & -\frac{1}{14} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = -\frac{6}{35} \begin{pmatrix} a \\ b \end{pmatrix} \quad (31)$$

for λ_2. This has solution $a = -1$ and $b = 1$. Thus, the general solution is

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = c_1 \begin{pmatrix} 10 \\ 14 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-\frac{6}{35}t} \quad (32)$$

For part (iii), matching with $y_1(0) = 0$ and $y_2(0) = 2$, this gives $c_1 = 1/12$ and $c_2 = 5/6$ and hence

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \frac{1}{12} \begin{pmatrix} 10 \\ 14 \end{pmatrix} + \frac{5}{6} \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-\frac{6}{35}t} \quad (33)$$

By the way, not the end result, clearly the exponetially decaying part goes away with time so that

$$\lim_{t\to\infty} \mathbf{y} = \begin{pmatrix} \frac{5}{6} \\ \frac{7}{6} \end{pmatrix} \quad (34)$$