2E2 Tutorial Sheet 7 First Term[1]

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1. (2) Use the Z-tranform to solve the difference equation

$$x_{k+2} - 8x_{k+1} + 15x_k = 1 \tag{1}$$

with $x_1 = 0$ and $x_0 = 0$.

Solution: Start by taking the Z-transform of both sides. Writing $\mathcal{Z}[(x_k)] = X(z)$ and using $x_0 = x_1 = 0$ we have

$$z^2X - 8zX + 15X = \mathcal{Z}[(1)] = \frac{z}{z-1} \tag{2}$$

Since $z^2 - 8z + 15 = (z-3)(z-5)$ we have

$$X = \frac{z}{(z-1)(z-3)(z-5)} \tag{3}$$

or

$$\frac{1}{z}X = \frac{1}{(z-1)(z-3)(z-5)} \tag{4}$$

Now, partial fractions:

$$\frac{1}{(z-1)(z-3)(z-5)} = \frac{A}{z-1} + \frac{B}{z-3} + \frac{C}{z-5} \tag{5}$$

so

$$1 = A(z-3)(z-5) + B(z-1)(z-5) + C(z-1)(z-3) \tag{6}$$

Next, $z = 1$ gives $A = 1/8$, $z = 3$ gives $B = -1/4$ and $z = 5$ gives $C = 1/8$. Hence

$$\frac{1}{(z-1)(z-3)(z-5)} = \frac{1}{8(z-1)} - \frac{1}{4(z-3)} + \frac{1}{8(z-5)} \tag{7}$$

giving

$$X = \frac{z}{8(z-1)} - \frac{z}{4(z-3)} + \frac{z}{8(z-5)} \tag{8}$$

Inverting the Z-tranform gives

$$x_k = \frac{1}{8} - \frac{1}{4}3^k + \frac{1}{8}5^k \tag{9}$$

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2. (2) Use the Z-tranform to solve the difference equation

$$x_{k+2} - 8x_{k+1} + 15x_k = 3^k \tag{10}$$

with $x_1 = 0$ and $x_0 = 0$.

Solution: Again, take the Z-tranform of both sides

$$z^2X - 8zX + 15X = \mathcal{Z}[(3^k)] = \frac{z}{z-3} \tag{11}$$

so

$$\frac{1}{z}X = \frac{1}{(z-3)^2(z-5)} \tag{12}$$

We need to do a partial fraction expansion with a repeated root:

$$\frac{1}{(z-3)^2(z-5)} = \frac{A}{(z-3)^2} + \frac{B}{z-3} + \frac{C}{z-5} \tag{13}$$

and so

$$1 = A(z-5) + B(z-3)(z-5) + C(z-3)^2 \tag{14}$$

Choose $z = 3$ to find $A = -1/2$, $z = 5$ to get $C = 1/4$ and then substitute $z = 0$ to work out B by putting in the known values of A and C:

$$1 = -\frac{1}{2}(-5) + 15B + \frac{1}{4}9 \tag{15}$$

Solving this gives $B = -1/4$. This means that

$$X = -\frac{z}{2(z-3)^2} - \frac{z}{4(z-3)} + \frac{z}{4(z-5)} \tag{16}$$

To invert we need to recall the table entry:

$$\mathcal{Z}[(kr^{k-1})] = \frac{z}{(z-r)^2} \tag{17}$$

We get

$$x_k = -\frac{1}{2}k3^{k-1} - \frac{1}{4}3^k + \frac{1}{4}5^k \tag{18}$$

3. (2) Use the Z-tranform to solve the difference equation

$$x_{k+2} - 8x_{k+1} + 15x_k = \delta_k \tag{19}$$

with $x_1 = 0$ and $x_0 = 0$. Remember δ_k is the unit pulse with $\delta_k = (1, 0, 0, 0, \ldots)$.

Solution: Take the Z-tranform of both sides,

$$z^2X - 8zX + 15X = \mathcal{Z}[(\delta_k)] = 1 \tag{20}$$

so

$$X = \frac{1}{(z-3)(z-5)} \tag{21}$$

Now, do partial fractions:

$$\frac{1}{(z-3)(z-5)} = \frac{A}{z-3} + \frac{B}{z-5} \tag{22}$$

and so

$$1 = A(z-5) + B(z-3) \tag{23}$$

Choose $z = 3$ for $A = -1/2$ and $z = 5$ for $B = 1/2$ and we obtain

$$X = -\frac{1}{2(z-3)} + \frac{1}{2(z-5)} \tag{24}$$

Now, the problem is that there are no z's on top, if there were it would be easy:

$$-\frac{1}{2(z-3)} + \frac{1}{2(z-5)} = \mathcal{Z}\left[\left(-\frac{1}{2}3^k + \frac{1}{2}5^k\right)_{k=0}^{\infty}\right] \tag{25}$$

In other words

$$X = \frac{1}{z}\mathcal{Z}\left[-\frac{1}{2}3^k + \frac{1}{2}5^k\right] \tag{26}$$

Now, this is what we get from the delay theorem, x_k is the delay of the sequence by one step, so,

$$x_k = \begin{cases} 0 & k = 0 \\ -\frac{1}{2}3^{k-1} + \frac{1}{2}5^{k-1} & k > 0 \end{cases} \tag{27}$$

4. (2) Use the Z-tranform to solve the difference equation

$$x_{k+2} - 8x_{k+1} + 15x_k = 0 \tag{28}$$

with $x_1 = 2$ and $x_0 = 3$.

Solution: The complication this time is provided by the initial conditions, remember that $\mathcal{Z}[(x_{k+2})] = z^2X - z^2x_0 - zx_1$ and that $\mathcal{Z}[(x_{k+1})] = zX - zx_0$:

$$z^2X - 3z^2 - 2z - 8zX + 24z + 15X = 0 \tag{29}$$

or

$$X = \frac{3z^2 - 22z}{(z-3)(z-5)} \tag{30}$$

It is convenient to bring one z over to the right:

$$\frac{1}{z}X = \frac{3z-22}{(z-3)(z-5)} \tag{31}$$

and now do partial fractions:

$$\frac{3z-22}{(z-3)(z-5)} = \frac{A}{z-3} + \frac{B}{z-5} \tag{32}$$

so

$$3z - 22 = A(z-5) + B(z-3) \tag{33}$$

Hence $z = 3$ gives $-13 = -2A$ or $A = 13/2$ and $z = 5$ gives $-7 = 2B$ so $B = -2/7$, or,

$$X = \frac{13z}{2(z-3)} - \frac{7z}{2(z-5)} \tag{34}$$

and hence

$$x_k = \frac{13}{2}3^k - \frac{7}{2}5^k \tag{35}$$