1. (2) Use the Z-transform to solve the difference equation
\[ x_{k+2} - 8x_{k+1} + 15x_k = 1 \]  
with \( x_1 = 0 \) and \( x_0 = 0 \).

Solution: Start by taking the Z-transform of both sides. Writing
\[ Z[(x_k)] = X(z) \]
and so
\[ 1 \]
we have
\[ 1 = A(z - 5) + B(z - 1)(z - 5) + C(z - 1)(z - 3) \]  
Next, \( z = 1 \) gives \( A = 1/8 \), \( z = 3 \) gives \( B = -1/4 \) and \( z = 5 \) gives \( C = 1/4 \). Hence
\[ \frac{1}{(z - 1)(z - 3)(z - 5)} = \frac{A}{z - 1} + \frac{B}{z - 3} + \frac{C}{z - 5} \]  
so
\[ 1 = A(z - 3)(z - 5) + B(z - 1)(z - 5) + C(z - 1)(z - 3) \]  
Choosing \( z = 3 \) to find \( A = -1/2 \), \( z = 5 \) to get \( C = 1/4 \) and then substitute \( z = 0 \) to work out \( B \) by putting in the known values of \( A \) and \( C \):
\[ 1 = -\frac{1}{2}(-5) + 15B + \frac{1}{4}9 \]  
Solving this gives \( B = -1/4 \). This means that
\[ X = \frac{z}{2(z - 3)^2} - \frac{z}{4(z - 3)^2} + \frac{z}{4(z - 5)} \]  
To invert we need to recall the table entry:
\[ Z[(kr^{k-1})] = \frac{z}{(z - k)^2} \]  
We get
\[ x_k = -\frac{1}{2}k3^{k-1} - \frac{1}{4}3^k + \frac{1}{4}5^k \]

2. (2) Use the Z-transform to solve the difference equation
\[ x_{k+2} - 8x_{k+1} + 15x_k = 3^k \]  
with \( x_1 = 0 \) and \( x_0 = 0 \).

Solution: Again, take the Z-transform of both sides
\[ z^2X - 8zX + 15X = Z[(3^k)] = \frac{z}{z - 3} \]  
so
\[ \frac{1}{z}X = \frac{1}{(z - 3)^2(z - 5)} \]  
We need to do a partial fraction expansion with a repeated root:
\[ \frac{1}{(z - 3)^2(z - 5)} = \frac{A}{z - 3} + \frac{B}{z - 3} + \frac{C}{z - 5} \]  
and so
\[ 1 = A(z - 5) + B(z - 3)(z - 5) + C(z - 3)^2 \]  
Choose \( z = 3 \) to find \( A = -1/2, z = 5 \) to get \( C = 1/4 \) and then substitute \( z = 0 \) to work out \( B \) by putting in the known values of \( A \) and \( C \):
\[ 1 = -\frac{1}{2}(-5) + 15B + \frac{1}{4}9 \]  
Solving this gives \( B = -1/4 \). This means that
\[ X = -\frac{z}{2(z - 3)^2} - \frac{z}{4(z - 3)^2} + \frac{z}{4(z - 5)} \]  
To invert we need to recall the table entry:
\[ Z[(kr^{k-1})] = \frac{z}{(z - k)^2} \]  
We get
\[ x_k = -\frac{1}{2}k3^{k-1} - \frac{1}{4}3^k + \frac{1}{4}5^k \]

3. (2) Use the Z-transform to solve the difference equation
\[ x_{k+2} - 8x_{k+1} + 15x_k = \delta_k \]  
with \( x_1 = 0 \) and \( x_0 = 0 \). Remember \( \delta_k \) is the unit pulse with \( \delta_k = (1, 0, 0, \ldots) \).

Solution: Take the Z-transform of both sides,
\[ z^2X - 8zX + 15X = Z[(\delta_k)] = 1 \]  
2.\footnote{Conor Houghton, houghton@maths.tcd.ie, see also http://www.maths.tcd.ie/~houghton/2E2.html}
so

\[ X = \frac{1}{(z-3)(z-5)} \]  \hspace{1cm} (21)

Now, do partial fractions:

\[ \frac{1}{(z-3)(z-5)} = \frac{A}{z-3} + \frac{B}{z-5} \] \hspace{1cm} (22)

and so

\[ 1 = A(z-5) + B(z-3) \] \hspace{1cm} (23)

Choose \( z = 3 \) for \( A = -1/2 \) and \( z = 5 \) for \( B = 1/2 \) and we obtain

\[ X = -\frac{1}{2(z-3)} + \frac{1}{2(z-5)} \] \hspace{1cm} (24)

Now, the problem is that there are no \( z \)’s on top, if there were it would be easy:

\[ -\frac{1}{2(z-3)} + \frac{1}{2(z-5)} = Z \left[ \left( -\frac{1}{2}3^k + \frac{1}{2}5^k \right)_{k=0}^\infty \right] \] \hspace{1cm} (25)

In other words

\[ X = \frac{1}{2}Z \left[ -\frac{1}{2}3^k + \frac{1}{2}5^k \right] \] \hspace{1cm} (26)

Now, this is what we get from the delay theorem, \( x_k \) is the delay of the sequence by one step, so,

\[ x_k = \begin{cases} 0 & k = 0 \\ -\frac{1}{2}3^{k-1} + \frac{1}{2}5^{k-1} & k > 0 \end{cases} \] \hspace{1cm} (27)

4. (2) Use the \( Z \)-transform to solve the difference equation

\[ x_{k+2} - 8x_{k+1} + 15x_k = 0 \] \hspace{1cm} (28)

with \( x_1 = 2 \) and \( x_0 = 3 \).

\textit{Solution:} The complication this time is provided by the initial conditions, remember that \( Z[(x_{k+2})] = z^2X - z^2x_0 - zx_1 \) and that \( Z[(x_{k+1})] = zX - x_0 \): \n
\[ z^2X - 3z^2 - 2z - 8zX + 24z + 15X = 0 \] \hspace{1cm} (29)

or

\[ X = \frac{3z^2 - 22z}{(z-3)(z-5)} \] \hspace{1cm} (30)

It is convenient to bring one \( z \) over to the right:

\[ \frac{1}{z}X = \frac{3z - 22}{(z-3)(z-5)} \] \hspace{1cm} (31)

and now do partial fractions:

\[ \frac{3z - 22}{(z-3)(z-5)} = \frac{A}{z-3} + \frac{B}{z-5} \] \hspace{1cm} (32)

so

\[ 3z - 22 = A(z-5) + B(z-3) \] \hspace{1cm} (33)

Hence \( z = 3 \) gives \(-13 = -2A\) or \( A = 13/2 \) and \( z = 5 \) gives \(-7 = 2B \) so \( B = -7/2 \) or,

\[ X = \frac{13z}{2(z-3)} - \frac{7z}{2(z-5)} \] \hspace{1cm} (34)

and hence

\[ x_k = \frac{13}{2}3^k - \frac{7}{2}5^k \] \hspace{1cm} (35)