2E2 Tutorial Sheet 5 First Term, Solutions[1]

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1. (2) Find the convolution $(f * g)(t)$ when $f(t) = t$, $g(t) = e^{2t}$ $(t \geq 0)$.

   Solution: From the definition of convolutions

$$\begin{aligned}
(f * g)(t) &= \int_0^t f(\tau)g(t-\tau)\,d\tau = \int_0^t \tau e^{2(t-\tau)}\,d\tau \\
&= \int_0^t \tau e^{2t} e^{-2\tau}\,d\tau = e^{2t}\int_0^t \tau e^{-2\tau}\,d\tau \\
\text{Use integration by parts with} \quad & U = \tau, \quad dV = e^{-2\tau}\,d\tau \\
& dU = d\tau, \quad V = -\frac{1}{2}e^{-2\tau} \\
&= e^{2t}\int_0^t U\,dV = e^{2t}\left([UV]_0^t - \int_0^t V\,dU\right) \\
&= e^{2t}\left(\left[-\frac{\tau}{2}e^{-2\tau}\right]_0^t - \int_0^t -\frac{1}{2}e^{-2\tau}\,d\tau\right) \\
&= e^{2t}\left(-\frac{t}{2}e^{-2t} + 0 + \frac{1}{2}\int_0^t e^{-2\tau}\,d\tau\right) \\
&= -\frac{t}{2} + \frac{e^{2t}}{2}\left[-\frac{1}{2}e^{-2\tau}\right]_0^t \\
&= -\frac{t}{2} + \frac{e^{2t}}{2}\left(-\frac{1}{2}e^{-2t} + \frac{1}{2}\right) \\
&= -\frac{t}{2} - \frac{1}{4} + \frac{1}{4}e^{2t}
\end{aligned}$$

2. (2) Use the convolution theorem to find the function $f(t)$ with

$$\mathcal{L}(f) = \frac{1}{s^2(s-4)}. \qquad (1)$$

   Solution: We know $\mathcal{L}(t)) = \frac{1}{s^2}$ and $\mathcal{L}(e^{4t}) = \frac{1}{s-4}$. From the convolution theorem, we see

$$\mathcal{L}(f) = \frac{1}{s^2(s-4)} = \mathcal{L}(t)\mathcal{L}(e^{4t}) = \mathcal{L}(t * e^{4t})$$

   so that $f(t)$ is the convolution $t * e^{4t}$.

$$f(t) = \int_0^t \tau e^{4(t-\tau)}\,d\tau$$

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$$= \int_0^t \tau e^{4t} e^{-4\tau}\, d\tau = e^{4t} \int_0^t \tau e^{-4\tau}\, d\tau$$

Use integration by parts with $\quad U = \tau, \quad dV = e^{-4\tau}\, d\tau$

$$dU = d\tau, \quad V = -\frac{1}{4} e^{-4\tau}$$

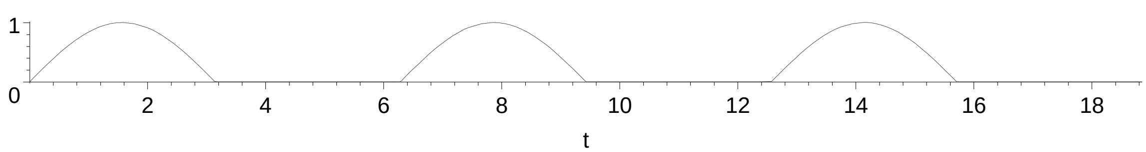
$$\begin{aligned}
&= e^{4t} \int_0^t U\, dV = e^{4t} \left([UV]_0^t - \int_0^t V\, dU \right) \\
&= e^{4t} \left(\left[-\frac{\tau}{4} e^{-4\tau} \right]_0^t - \int_0^t -\frac{1}{4} e^{-4\tau}\, d\tau \right) \\
&= e^{4t} \left(-\frac{t}{4} e^{-4t} + 0 + \frac{1}{4} \int_0^t e^{-4\tau}\, d\tau \right) \\
&= -\frac{t}{4} + \frac{e^{4t}}{4} \left[-\frac{1}{2} e^{-4\tau} \right]_0^t \\
&= -\frac{t}{4} + \frac{e^{4t}}{4} \left(-\frac{1}{4} e^{-4t} + \frac{1}{4} \right) \\
&= -\frac{t}{4} - \frac{1}{16} + \frac{1}{16} e^{4t}
\end{aligned}$$

3. (4) Use the formula for the Laplace transform of a periodic function with period c:

$$\mathcal{L}(f) = \frac{1}{1 - e^{-cs}} \int_0^c f(t) e^{-st} dt \qquad (2)$$

to find the Laplace transform of a half-rectified wave

$$f(t) = \begin{cases} \sin t & \sin t > 0 \\ 0 & \sin t \leq 0 \end{cases} \qquad (3)$$



This is the form a AC current has after going through a diode.

So we substitute this into the formula

$$\mathcal{L}(f) = \frac{1}{1 - e^{-2\pi s}} \int_0^{2\pi} f(t) e^{-st} dt = \frac{1}{1 - e^{-2\pi s}} \int_0^{\pi} \sin t e^{-st} dt \qquad (4)$$

We need to do the integral. There are two obvious ways, the first is to split the sine into exponentials

$$\begin{aligned}
\int_0^{\pi} \sin t e^{-st} dt &= \frac{1}{2i} \left(\int_0^{\pi} e^{(i-s)t} dt - \int_0^{\pi} e^{-(i+s)t} dt \right) \\
&= \frac{1}{2i} \left[\frac{1}{i-s} \left(e^{(i-s)\pi} - 1 \right) + \frac{1}{i+s} \left(e^{-(i+s)\pi} - 1 \right) \right] \qquad (5)
\end{aligned}$$

Now, we use

$$e^{i\pi} = e^{-i\pi} = -1 \tag{6}$$

and

$$\begin{aligned}
\frac{1}{i-s} &= \frac{1}{i-s}\frac{-i-s}{-i-s} = -\frac{s+i}{s^2+1} \\
\frac{1}{i+s} &= \frac{1}{i+s}\frac{-i+s}{-i+s} = \frac{s-i}{s^2+1}
\end{aligned} \tag{7}$$

to get

$$\int_0^\pi \sin t e^{-st} dt = \frac{1+e^{-s\pi}}{1+s^2} \tag{8}$$

or

$$\mathcal{L}(f) = \frac{1}{s^2+1}\frac{1+e^{-s\pi}}{1-e^{-2s\pi}} = \frac{1}{s^2+1}\frac{1}{1-e^{-s\pi}} \tag{9}$$

where the final equality uses

$$1 - e^{-2s\pi} = \left(1 - e^{-s\pi}\right)\left(1 + e^{-s\pi}\right) \tag{10}$$

The other way to do the integral is to integrate by parts. Briefly, write

$$\begin{aligned}
I = \int_0^\pi \sin t e^{-st} dt &= -\frac{1}{s}\int_0^\pi \cos t e^{-st} dt \\
&= -\frac{1}{s}\left[-\frac{1}{s}\left(e^{-\pi s}+1\right) + \frac{1}{s} I\right]
\end{aligned} \tag{11}$$

and solve for I to get the answer given at (8) above.