1. (2) Use Laplace transform methods to solve the differential equation

\[ f'' + 2f' - 3f = \begin{cases} 1, & 0 \leq t < c \\ 0, & t \geq c \end{cases} \quad (1) \]

subject to the initial conditions \( f(0) = f'(0) = 0 \). (3)

**Solution:** Taking Laplace transforms of both sides and using the tables for the Laplace transform of the right hand side function, leads to

\[
(s^2 + 2s - 3)F = \frac{1 - e^{-cs}}{s} \\
F = \frac{1 - e^{-cs}}{s(s^2 + 2s - 3)} \\
= \frac{1}{s(s - 1)(s + 3)} \\
= (1 - e^{-cs}) \left( \frac{A}{s} + \frac{B}{s - 1} + \frac{C}{s + 3} \right) \quad (2)
\]

Concentrating on the partial fractions part, we have

\[
\frac{1}{s(s - 1)(s + 3)} = \frac{A}{s} + \frac{B}{s - 1} + \frac{C}{s + 3} \\
1 = A(s - 1)(s + 3) + Bs(s + 3) + Cs(s - 1) \\
\]

Concentrating on the partial fractions part, we have

\[
s = 0: \\
1 = -3A \\
A = -\frac{1}{3} \\
s = 1: \\
1 = 0 + 4B + 0 \\
B = \frac{1}{4} \\
s = -3: \\
1 = 0 + 012C \\
C = \frac{1}{12}
\]

Hence we have

\[
F = (1 - e^{-cs}) \left( -\frac{1}{3} \frac{1}{s} + \frac{1}{4} \frac{1}{s - 1} + \frac{1}{12} \frac{1}{s + 3} \right) \quad (3)
\]

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From the tables, we know that
\[ \mathcal{L}\left(-\frac{1}{3} + \frac{1}{4}e^t - \frac{1}{12}e^{-3t}\right) = -\frac{1}{3s} + \frac{1}{4s-1} + \frac{1}{12s+3} \quad (4) \]
and then using the second shift theorem
\[ f(t) = -\frac{1}{3} + \frac{1}{4}e^t + \frac{1}{12}e^{-3t} - H_c(t)\left(-\frac{1}{3} + \frac{1}{4}(t-c) + \frac{1}{12}e^{-3(t-c)}\right) \quad (5) \]

2. (3) Use Laplace transform methods to solve the differential equation
\[ f'' + 2f' - 3f = \begin{cases} 0, & 0 \leq t < 1 \\ 1, & 1 \leq t < 2 \\ 0, & t \geq 2 \end{cases} \quad (6) \]
subject to the initial conditions \( f(0) = 0 \) and \( f'(0) = 0 \).

Solution: So the thing here is to rewrite the right hand side of the equations in terms of Heaviside functions. Remember the definition of the Heaviside function:
\[ H_a(t) = \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases} \quad (7) \]
so the Heaviside function is zero until \( a \) and then it is one. The right hand side is zero until \( t = 1 \) and then it is one until \( t = 2 \) and then it is zero again. Consider \( H_1(t) - H_2(t) \), this is zero until you reach \( t = 1 \), then the first Heaviside function switches on, the other one remains zero. Things stay like this until you reach \( t = 2 \), then the second Heaviside function switches on as well and you get \( 1 - 1 = 0 \). Thus
\[ H_1(t) - H_2(t) = \begin{cases} 0, & 0 \leq t < 1 \\ 1, & 1 \leq t < 2 \\ 0, & t \geq 2 \end{cases} \quad (8) \]
Now, using
\[ \mathcal{L}(H_a(t)) = \frac{e^{-as}}{s} \quad (9) \]
we take the Laplace transform of the differential equation:
\[ s^2F + 2sF - 3F = \frac{e^{-s}}{s} - \frac{e^{-2s}}{s} \quad (10) \]
This gives
\[ (s^2 + 2s - 3)F = \frac{1}{s}(e^{-s} - e^{-2s}) \]
\[ F = \frac{1}{s(s-1)(s+3)}(e^{-s} - e^{-2s}) \quad (11) \]
Now, if you look at the soln to problem sheet 4, question 3 you’ll see that

\[
\frac{1}{s(s - 1)(s + 3)} = \frac{1}{3s} + \frac{1}{4(s - 1)} + \frac{1}{12(s + 3)}
\]  

(12)

and we know that

\[
\mathcal{L} \left( \frac{1}{3} + \frac{1}{4} e^t + \frac{1}{12} e^{-3t} \right) = \frac{1}{3} + \frac{1}{4(s - 1)} + \frac{1}{12(s + 3)}
\]  

(13)

In other word, if it wasn’t for the exponentials we’d know the little f. However, we know from the second shift thereom that the affect of the exponential \(e^{-as}\) is to change \(t\) to \(t - a\) and to introduce an overall factor of \(H_a(t)\). Thus

\[
f = H_1(t) \left( \frac{1}{3} + \frac{1}{4} e^{t-1} + \frac{1}{12} e^{-3(t-3)} \right) - H_2(t) \left( \frac{1}{3} + \frac{1}{4} e^{t-2} + \frac{1}{12} e^{-3(t-6)} \right)
\]  

(14)

3. (3) Use Laplace transform methods to solve the differential equation

\[
f'' + 2f' - 3f = \delta(t - 1)
\]  

subject to the initial conditions \(f(0) = 0\) and \(f'(0) = 1\).

**Solution:** The only thing that is unusual is that there is a delta function. We take the Laplace transform using

\[
\mathcal{L}(\delta(t - a)) = e^{-as}
\]  

(16)

hence

\[
(s^2 + 2s - 3)F - 1 = e^{-s}
\]  

(17)

Now, if we do partial fractions on \(1/(s^2 + 2s - 3)\) we get

\[
\frac{1}{s^2 + 2s - 3} = -\frac{1}{4(s + 3)} + \frac{1}{4(s - 1)}
\]  

(18)

Hence

\[
F = \left( -\frac{1}{4(s + 3)} + \frac{1}{4(s - 1)} \right) (1 + e^{-s})
\]  

(19)

Since

\[
\mathcal{L} \left( \frac{1}{4} e^{-3t} + \frac{1}{4} e^t \right) = -\frac{1}{4(s + 3)} + \frac{1}{4(s - 1)}
\]  

(20)

then, by the second shift theorem we have

\[
f = \left( -\frac{1}{4} e^{-3t} + \frac{1}{4} e^t \right) + H_1(t) \left( -\frac{1}{4} e^{-3(t-3)} + \frac{1}{4} e^{t-1} \right)
\]  

(21)