

2E2 Tutorial Sheet 4 First Term, Solutions¹

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1. (2) Use Laplace transform methods to solve the differential equation

$$f'' + 2f' - 3f = \begin{cases} 1, & 0 \leq t < c \\ 0, & t \geq c \end{cases} \quad (1)$$

subject to the initial conditions $f(0) = f'(0) = 0$. (3)

Solution: Taking Laplace transforms of both sides and using the tables for the Laplace transform of the right hand side function, leads to

$$\begin{aligned} (s^2 + 2s - 3)F &= \frac{1 - e^{-cs}}{s} \\ F &= \frac{1 - e^{-cs}}{s(s^2 + 2s - 3)} \\ &= (1 - e^{-cs}) \frac{1}{s(s-1)(s+3)} \\ &= (1 - e^{-cs}) \left(\frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+3} \right) \end{aligned} \quad (2)$$

Concentrating on the partial fractions part, we have

$$\begin{aligned} \frac{1}{s(s-1)(s+3)} &= \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+3} \\ 1 &= A(s-1)(s+3) + Bs(s+3) + Cs(s-1) \\ \underline{s=0:} \\ 1 &= -3A \\ A &= -\frac{1}{3} \\ \underline{s=1:} \\ 1 &= 0 + 4B + 0 \\ B &= \frac{1}{4} \\ \underline{s=-3:} \\ 1 &= 0 + 0 + 12C \\ C &= \frac{1}{12} \end{aligned}$$

Hence we have

$$F = (1 - e^{-cs}) \left(-\frac{1}{3} \frac{1}{s} + \frac{1}{4} \frac{1}{s-1} + \frac{1}{12} \frac{1}{s+3} \right) \quad (3)$$

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From the tables, we know that

$$\mathcal{L}\left(-\frac{1}{3} + \frac{1}{4}e^t - \frac{1}{12}e^{-3t}\right) = -\frac{1}{3}\frac{1}{s} + \frac{1}{4}\frac{1}{s-1} + \frac{1}{12}\frac{1}{s+3} \quad (4)$$

and then using the second shift theorem

$$f(t) = -\frac{1}{3} + \frac{1}{4}e^t + \frac{1}{12}e^{-3t} - H_c(t) \left(-\frac{1}{3} + \frac{1}{4}e^{(t-c)} + \frac{1}{12}e^{-3(t-c)}\right) \quad (5)$$

2. (3) Use Laplace transform methods to solve the differential equation

$$f'' + 2f' - 3f = \begin{cases} 0, & 0 \leq t < 1 \\ 1, & 1 \leq t < 2 \\ 0, & t \geq 2 \end{cases} \quad (6)$$

subject to the initial conditions $f(0) = 0$ and $f'(0) = 0$.

Solution: So the thing here is to rewrite the right hand side of the equations in terms of Heaviside functions. Remember the definition of the Heaviside function:

$$H_a(t) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases} \quad (7)$$

so the Heaviside function is zero until a and then it is one. The right hand side is zero until $t = 1$ and then it is one until $t = 2$ and then it is zero again. Consider $H_1(t) - H_2(t)$, this is zero until you reach $t = 1$, then the first Heaviside function switches on, the other one remains zero. Things stay like this until you reach $t = 2$, then the second Heaviside function switches on as well and you get $1 - 1 = 0$. Thus

$$H_1(t) - H_2(t) = \begin{cases} 0, & 0 \leq t < 1 \\ 1, & 1 \leq t < 2 \\ 0, & t \geq 2 \end{cases} \quad (8)$$

Now, using

$$\mathcal{L}(H_a(t)) = \frac{e^{-as}}{s} \quad (9)$$

we take the Laplace transform of the differential equation:

$$s^2F + 2sF - 3F = \frac{e^{-s}}{s} - \frac{e^{-2s}}{s} \quad (10)$$

This gives

$$\begin{aligned} (s^2 + 2s - 3)F &= \frac{1}{s}(e^{-s} - e^{-2s}) \\ F &= \frac{1}{s(s-1)(s+3)}(e^{-s} - e^{-2s}) \end{aligned} \quad (11)$$

Now, if you look at the soln to problem sheet 4, question 3 you'll see that

$$\frac{1}{s(s-1)(s+3)} = -\frac{1}{3s} + \frac{1}{4(s-1)} + \frac{1}{12(s+3)} \quad (12)$$

and we know that

$$\mathcal{L}\left(-\frac{1}{3} + \frac{1}{4}e^t + \frac{1}{12}e^{-3t}\right) = -\frac{1}{3s} + \frac{1}{4(s-1)} + \frac{1}{12(s+3)} \quad (13)$$

In other word, if it wasn't for the exponentials we'd know the little f. However, we know from the second shift theorem that the affect of the exponential e^{-as} is to change t to $t - a$ and to introduce an overall factor of $H_a(t)$. Thus

$$f = H_1(t) \left(-\frac{1}{3} + \frac{1}{4}e^{t-1} + \frac{1}{12}e^{-3t+3}\right) - H_2(t) \left(-\frac{1}{3} + \frac{1}{4}e^{t-2} + \frac{1}{12}e^{-3t+6}\right) \quad (14)$$

3. (3) Use Laplace transform methods to solve the differential equation

$$f'' + 2f' - 3f = \delta(t - 1) \quad (15)$$

subject to the initial conditions $f(0) = 0$ and $f'(0) = 1$.

Solution: The only thing that is unusual is that there is a delta function. We take the Laplace transform using

$$\mathcal{L}(\delta(t - a)) = e^{-as} \quad (16)$$

hence

$$(s^2 + 2s - 3)F - 1 = e^{-s} \quad (17)$$

Now, if we do partial fractions on $1/(s^2 + 2s - 3)$ we get

$$\frac{1}{s^2 + 2s - 3} = -\frac{1}{4(s+3)} + \frac{1}{4(s-1)} \quad (18)$$

Hence

$$F = \left(-\frac{1}{4(s+3)} + \frac{1}{4(s-1)}\right) (1 + e^{-s}) \quad (19)$$

Since

$$\mathcal{L}\left(-\frac{1}{4}e^{-3t} + \frac{1}{4}e^t\right) = -\frac{1}{4(s+3)} + \frac{1}{4(s-1)} \quad (20)$$

then, by the second shift theorem we have

$$f = \left(-\frac{1}{4}e^{-3t} + \frac{1}{4}e^t\right) + H_1(t) \left(-\frac{1}{4}e^{-3t+3} + \frac{1}{4}e^{t-1}\right) \quad (21)$$