2E2 Tutorial Sheet 3 First Term, Solutions

24 October 2003

1. Using the Laplace transform solve the differential equation

\[ f'' + f' - 6f = e^{-3t} \]  

with boundary conditions \( f(0) = f'(0) = 0 \).

Solution: So, as before, the subsidiary equation is

\[ s^2 F + sF - 6F = \frac{1}{s + 3} \]  

or

\[ F = \frac{1}{(s + 3)^2(s - 2)} \]  

As before, we do partial fractions

\[ \frac{1}{(s + 3)^2(s - 2)} = \frac{A}{s + 3} + \frac{B}{(s + 3)^2} + \frac{C}{s - 2} \]  

\( s = -3 \) gives \( B = -1/5 \) and \( s = 2 \) gives \( C = 1/25 \). Putting in \( s = 1 \) we find

\[ 1 = -4A + \frac{1}{5} + \frac{16}{25} \]  

and so \( A = -1/25 \). Putting all this together says that

\[ f = -\frac{1}{25}e^{-3t} - \frac{4}{5}e^{-3t} + \frac{1}{25}e^{2t} \]  

2. Using the Laplace transform solve the differential equation

\[ f'' + 6f' + 13f = 0 \]  

with boundary conditions \( f(0) = 0 \) and \( f'(0) = 1 \).

Solution: So, taking the Laplace transform of the equation we get,

\[ s^2 F - 1 + 6sF + 13F = 0 \]  

and, hence,

\[ F = \frac{1}{s^2 + 6s + 13} \]  

Now, using minus b plus or minus the square root of b squared minus four a c all over two a, we get

\[ s^2 + 6s + 13 = 0 \]  

\[ s = -3 \pm \sqrt{36 - 52} \]  

which means

\[ s^2 + 6s + 13 = (s + 3 - 2i)(s + 3 + 2i) \]  

Next, we do the partial fraction expansion,

\[ \frac{1}{s^2 + 6s + 13} = \frac{A}{s + 3 - 2i} + \frac{B}{s + 3 + 2i} \]  

and multiplying across we get

\[ 1 = A(s + 3 + 2i) + B(s + 3 - 2i) \]  

therefore we choose \( s = -3 + 2i \) to get

\[ A = -\frac{1}{4i} = \frac{-i}{4} \]  

and \( s = -3 - 2i \) to get

\[ B = -\frac{1}{4i} = \frac{i}{4} \]  

and so

\[ F = -\frac{i}{4} \frac{1}{s + 3 - 2i} + \frac{i}{4} \frac{1}{s + 3 + 2i} \]  

If we take the inverse transform

\[ f = \frac{-i}{4} e^{-(3-2i)t} + \frac{i}{4} e^{-(3+2i)t} \]  

\[ = \frac{i}{4} e^{-3t}(e^{-2it} - e^{2it}) \]  

\[ = \frac{1}{2} e^{-3t} \sin 2t \]  

3. Using the Laplace transform solve the differential equation

\[ f'' + 6f' + 13f = e^t \]  

with boundary conditions \( f(0) = 0 \) and \( f'(0) = 0 \).

Solution: Taking the Laplace transform of the equation gives

\[ s^2 F + 6s F + 13F = \frac{1}{s - 1} \]  

\[ \]
so that

\[ F = \frac{1}{(s-1)(s+3+2i)(s+3-2i)}. \]  \hspace{1cm} (21)

We write

\[ \frac{1}{(s-1)(s+3+2i)(s+3-2i)} = \frac{A}{s+3-2i} + \frac{B}{s+3+2i} + \frac{C}{s-1} \]  \hspace{1cm} (22)

giving

\[ 1 = A(s-1)(s+3+2i) + B(s-1)(s+3-2i) + C(s+3-2i)(s+3+2i). \]  \hspace{1cm} (23)

When \( s = -3 + 2i \) gives

\[ 1 = A(-4+2i)(4i) = A(-8 - 16i) \]  \hspace{1cm} (24)

so

\[ A = \frac{1}{8 + 16i} = \frac{1}{8 + 16i} \frac{8 - 16i}{8 - 16i} = \frac{1 + 2i}{40} \]  \hspace{1cm} (25)

In the same way, \( s = -3 - 2i \) leads to

\[ B = -\frac{1 - 2i}{40} \]  \hspace{1cm} (26)

and, finally, \( s = 1 \) gives

\[ C = \frac{1}{20}. \]  \hspace{1cm} (27)

Putting all this together we get

\[ F = -\frac{1 + 2i}{40} \frac{1}{s + 3 - 2i} - \frac{1 - 2i}{40} \frac{1}{s + 3 + 2i} + \frac{1}{20} \frac{1}{s - 1} \]  \hspace{1cm} (28)

and so

\[ f = -\frac{1 + 2i}{40} e^{(3-2i)t} - \frac{1 - 2i}{40} e^{(3+2i)t} + \frac{1}{20} e^t \]
\[ = -\frac{1}{40} e^{-3t} [(1 + 2i)e^{2it} + (1 - 2i)e^{-2it}] + \frac{1}{20} e^t \]  \hspace{1cm} (29)

We then substitute in

\[ e^{2it} = \cos 2t + i \sin 2t \]
\[ e^{-2it} = \cos 2t - i \sin 2t \]  \hspace{1cm} (30)

to end up with

\[ f = \frac{1}{20} e^{-3t} [2 \sin 2t - \cos 2t] + \frac{1}{20} e^t \]  \hspace{1cm} (31)